Examples of Direct and Inverse Image Proofs

Several students were asking: what does a proof that $f(E)$ is a particular set look like, and similarly for $f^{-1}(H)$? Here is an example.

Example One:

Problem: Let f be the function with domain $\mathbb{R}$ and codomain $\mathbb{R}$ given by $f(x) = 1 + x^2$. Let

$$E = \{x \in \mathbb{R} : x < -3\} \cup \{x \in \mathbb{R} : x \geq 2\}.$$

Find $f(E)$ and $f^{-1}(H)$. Give a proof.

Solution # 1:

We claim that $f(E) = \{y : y \geq 5\}$. Proof: First assume that $y \in f(E)$. So there is some $x \in E$ such that $f(x) = y$. Either $x < -3$ or $x \geq 2$, so either $x^2 > 9$ or $x^2 \geq 4$. So $x^2 \geq 4$. So $y = 1+x^2 \geq 5$. Therefore $f(E) \subseteq \{y : y \geq 5\}$.

Now assume $y \geq 5$. Then $y - 1 \geq 4$, so $\sqrt{y-1} \geq 2$. So $\sqrt{y-1} \in E$. Since $f(\sqrt{y-1}) = 1 + (\sqrt{y-1})^2 = y$, $y \in f(E)$. Therefore $\{y : y \geq 5\} \subseteq f(E)$. This proves the claim.

You could also write the same argument more symbolically:

Solution # 2: $f(E) = \{y : y \geq 5\}$.

$\underline{f(E) \subset \{y : y \geq 5\}}$. Assume that $y \in f(E)$. There is an $x \in E$ such that $f(x) = y$.

$$\begin{aligned} x \in E &\Rightarrow x < -3 \text{ or } x \geq 2 \\ &\Rightarrow x^2 > 9 \text{ or } x^2 \geq 4 \\ &\Rightarrow 1 + x^2 > 10 \text{ or } 1 + x^2 \geq 5 \\ &\Rightarrow 1 + x^2 \geq 5. \end{aligned}$$

So $f(E) \subset \{y : y \geq 5\}$.

$\underline{\{y : y \geq 5 \subseteq f(E)\}}$: Assume that $y \in \{y : y \geq 5\}$.

$$\begin{aligned} y \geq 5 &\Rightarrow y - 1 \geq 4 \\ &\Rightarrow \sqrt{y-1} \geq 2 \\ &\Rightarrow \sqrt{y-1} \in E \\ &\Rightarrow y = 1 + (\sqrt{y-1})^2 = f(\sqrt{y-1}) \in f(E). \end{aligned}$$

So $\{y : y \geq 5\} \subseteq f(E)$.

Example Two:

Problem: Let $f(x) = 1 + x^2$ with domain $\mathbb{R}$ and codomain $\mathbb{R}$ as above and let

$$H = \{y \in \mathbb{R} : y > 10\}.$$

Find $f^{-1}(H)$. Give a proof.

Solution # 1:

We claim that $f^{-1}(H) = \{x : x < -3 \text{ or } x > 3\}$.

$\underline{f^{-1}(H) \subseteq \{x : x < -3 \text{ or } x > 3\}}$:

$$\begin{aligned} x \in f^{-1}(H) &\Rightarrow f(x) \in H \\ &\Rightarrow 1 + x^2 > 10 \\ &\Rightarrow x^2 > 9 \\ &\Rightarrow x > 3 \text{ or } x < -3. \end{aligned}$$

So $f^{-1}(H) \subseteq \{x : x < -3 \text{ or } x > 3\}$.

<u>$\{x : x < -3 \text{ or } x > 3\} \subseteq f^{-1}(H)$</u>:

$$\begin{aligned} x < -3 \text{ or } x > 3 &\Rightarrow x^2 > 9 \\ &\Rightarrow 1 + x^2 > 10 \\ &\Rightarrow f(x) \in H. \end{aligned}$$

So $\{x : x < -3 \text{ or } x > 3\} \subseteq f^{-1}(H)$.

Solution # 2:

We claim that $f^{-1}(H) = \{x : x < -3 \text{ or } x > 3\}$. Proof:

$$\begin{aligned} x \in f^{-1}(H) &\Leftrightarrow f(x) \in H \\ &\Leftrightarrow 1 + x^2 > 10 \\ &\Leftrightarrow x^2 > 9 \\ &\Leftrightarrow x > 3 \text{ or } x < -3. \end{aligned}$$

So $f^{-1}(H) = \{x : x < -3 \text{ or } x > 3\}$.

A few tips:

- You do NOT need to quote the real number properties for questions from Section 1 (they are not even covered yet in Section 1 of the text).
- You must prove both set inclusions. A common mistake is to only prove one inclusion.
- More than half of the class will make errors when using $\Leftrightarrow$. I recommend avoiding it at first.
- Even symbolic claims sound like complete sentences when read allowed. For example, "$x < -3$ or $x > 3 \Rightarrow x^2 > 9$" means "if x is less than -3 or x is greater than 3 then x squared is greater than nine. If you read it allowed and it doesn't sound grammatical, then probably you're making a mistake. The verb in a phrase is often "$=$", "$<$", "$\exists$" ("there exists"), and the like. "$\Rightarrow$" is not a verb.