The Definite Integral

MATH 1700

Readings

Section 5.2

The definition of area

Definition (Riemann)

Let f be a function on $[a, b]$. For some integer $n > 0$ let $x_0 = a$ and $x_i = a + i\Delta x$. Choose x_i^* such that $x_{i-1} \leq x_i^* \leq x_i$. Then the **integral of f over** $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists. If the limit exists f is called **integrable** over $[a, b]$.

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Actually this is not quite the precise definition, but it's good enough for this course.
Idea: in the limit, the approximation by rectangles goes to the actual net area!

When is f integrable?

This is a very hard question in general. For us, this is enough:

Theorem

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$.

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Theorem

If f is integrable on $[a, b]$, then we can use the limit of the left or right sum (or any other choice of points) to evaluate the integral. E.g. for $x_i = x_0 + i\Delta x$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

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You can use this to compute integrals of some functions, if you know the area of the shape already.

What if $a \geq b$?

If $a \geq b$, then we define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

(In some sense, this follows from the definition given earlier, since $\Delta x = (b-a)/n < 0$; we're just removing the implicit assumption that $b > a$.)

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If $a = b$ then it follows from the definition (since $\Delta x = 0$) that

$$\int_a^a f(x)dx = 0.$$

Properties of the Definite Integral

Assuming that the integrals below exist,

1. $\int_a^b c dx = c(b-a)$ for any constant c
2. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$
3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ for any constant c
4. $\int_a^b (f(x)-g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$

More properties

5. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$

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6. If $f(x) \geq 0$ for all $a \leq x \leq b$ then

$$\int_a^b f(x)dx > 0.$$

7. If $f(x) \geq g(x)$ for all $a \leq x \leq b$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

8. If $m \leq f(x) \leq M$ for all $a \leq x \leq b$ then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$