A functional-analytic proof of the conformal welding theorem

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Conformal welding theorem

Definition

A quasiconformal map $\phi : A \to B$ between open connected domains A and B in $\mathbb{C}$ is a homeomorphism such that

1. ϕ is absolutely continuous on almost every vertical and horizontal line in every closed rectangle $[a, b] \times [c, d] \subseteq A$
2. $$\left\| \frac{\overline{\partial} f}{\partial f} \right\|_\infty \leq k$$
   for some fixed $k < 1$.

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Definition

A quasisymmetric map $\phi : S^1 \to S^1$ is a homeomorphism which is the boundary values of some quasiconformal map $H : \mathbb{D} \to \mathbb{D}$.

Conformal welding theorem

Theorem (Conformal welding theorem)

Let $\phi : S^1 \to S^1$ be a quasisymmetric map and let $\alpha > 0$. There is a pair of maps $f : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D}^* \to \overline{\mathbb{C}}$ such that

1. *f is one-to-one and holomorphic, and has a quasiconformal extension to $\overline{C}$*
2. *g is one-to-one and holomorphic except for a simple pole at ∞, and has a quasiconformal extension to $\overline{C}$*
3. *$f(0) = 0$, $g(\infty) = \infty$ and $g'(\infty) = \alpha$.*
4. *$\phi = g^{-1} \circ f$ on S^1.*

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• standard proof uses existence and uniqueness to solutions of the Beltrami equation.

• We will give another proof using symplectic geometry and Grunsky inequalities.

The function spaces $\mathcal{H}$ and $\mathcal{H}_*$

Let $\mathcal{H}$ denote the space of L^2 functions h on S^1 such that

$$\sum_{n=-\infty}^{\infty} |n||\hat{h}(n)|^2 < \infty.$$

Define

$$\|h\|^2 = |\hat{h}(0)|^2 + \sum_{n=-\infty}^{\infty} |n||\hat{h}(n)|^2.$$

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We will also consider

$$\mathcal{H}_* = \{h \in \mathcal{H} \,:\, \hat{h}(0) = 0\}$$

with norm

$$\|h\|_*^2 = \sum_{n=-\infty}^{\infty} |n||\hat{h}(n)|^2.$$

Decomposition of $\mathcal{H}_*$

$$\mathcal{H}_+ = \{h \in \mathcal{H}_* \,:\, h = \sum_{n=1}^{\infty} h_n e^{in\theta}\}$$

$$\mathcal{H}_- = \{h \in \mathcal{H}_* \,:\, h = \sum_{n=-\infty}^{-1} h_n e^{in\theta}\}.$$

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It is well-known that we have the following isometries

$$\begin{aligned} \mathcal{H}_+ &\cong \mathcal{D}(\mathbb{D}) = \{h : \mathbb{D} \to \mathbb{C} \,:\, \iint_{\mathbb{D}} |h'|^2 \, dA < \infty \;\; h(0) = 0\} \\ \mathcal{H}_- &\cong \mathcal{D}(\mathbb{D}^*) = \{h : \mathbb{D}^* \to \mathbb{C} \,:\, \iint_{\mathbb{D}^*} |h'|^2 \, dA < \infty \;\; h(\infty) = 0\} \end{aligned}$$

Summarized in Nag and Sullivan.

Composition operators on $\mathcal{H}$ and $\mathcal{H}_*$

We consider two composition operators

$$C_\phi : \mathcal{H} \to \mathcal{H} \qquad C_\phi h = h \circ \phi$$

$$\hat{C}_\phi : \mathcal{H}_* \to \mathcal{H}_* \qquad C_\phi h = h \circ \phi - \frac{1}{2\pi} \int_{S^1} h \circ \phi(e^{i\theta})\, d\theta.$$

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Theorem (Nag and Sullivan, quoting notes of Zinsmeister)

$\hat{C}_\phi$ is bounded if ϕ is a quasisymmetry.

Theorem (S and Staubach)

If ϕ is a quasisymmetry then C_ϕ is bounded.

Sketch of a new proof

Treat ϕ as a composition operator C_ϕ on $\mathcal{H}$: we want to solve for unknown functions f and g in equation $f \circ \phi^{-1} = g$, $g_{-1} = \alpha$.

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Using the decomposition $\mathcal{H} = [\mathcal{H}_+] \oplus [\mathbb{C} \oplus \mathcal{H}_-]$, the welding equation can be written

$$C_\phi f = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}$$

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$$M_{++} f = g_+ \quad \text{and} \quad M_{+-} f = g_- .$$

where $g_+ = g_{-1} z = \alpha z$ and $g_- = g_0 + g_1/z + g_2/z^2 + \cdots$.

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which leads to the solution

$$f = M_{++}^{-1} g_+ \quad g_- = M_{+-} f.$$

What are the gaps?

We need to show that

- M_{++} is invertible: will use symplectic geometry and results of Nag and Sullivan, Takhtajan and Teo.
- The solutions in $\mathcal{H}$ so obtained have the desired properties: conformal with quasiconformal extensions: will use Grunsky inequalities.

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Here we go!

Symplectic structure on $\mathcal{H}_*$

For $f, g \in \mathcal{H}_*$ let

$$\omega(f,g) = -i \sum_{n=-\infty}^{\infty} f_n g_{-n}.$$

If one restricts to the real subspace (such that $\hat{f}(-n) = \overline{\hat{f}(n)}$) this is a non-degenerate anti-symmetric form $2\text{Im}\left(\sum_{n=1}^{\infty} \overline{\hat{f}(n)}\hat{g}(n)\right)$.

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Theorem (Nag and Sullivan)

If $\phi : S^1 \to S^1$ is quasisymmetric then $\hat{C}_\phi$ is a symplectomorphism (that is, $\omega(\hat{C}_\phi f, \hat{C}_\phi g) = \omega(f,g)$).

Note that $\hat{C}_\phi$ has the form

$$\begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}.$$

The infinite Siegel disc (Nag and Sullivan)

Definition

The infinite Siegel disc $\mathfrak{S}$ is the set of maps $Z : \mathcal{H}_- \to \mathcal{H}_+$ such that $Z^T = Z$ and $I - Z\overline{Z}$ is positive definite.

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Context:

- the graph of each Z is a Lagrangian subspaces of $\mathcal{H}_*$
- symplectomorphisms $\hat{C}_\phi$ act on them.

Definition

Let $\mathcal{L}$ be the set of bounded linear maps of the form

$$(P, Q) : \mathcal{H}_- \to \mathcal{H}_*$$

where $P : \mathcal{H}_- \to \mathcal{H}_+$ and $Q : \mathcal{H}_- \to \mathcal{H}_-$ are bounded operators satisfying $\overline{P}^T P - \overline{Q}^T Q > 0$ and $Q^T P = P^T Q$.

Two facts

- Q invertible $\Rightarrow PQ^{-1} \in \mathfrak{S} \Leftrightarrow (P, Q) \in \mathcal{L}$.
- $(P, Q)Q^{-1} = (PQ^{-1}, I)$ has the same image as (P, Q)

Invariance of $\mathcal{L}$

$\mathcal{L}$ is invariant under bounded symplectomorphisms.

Proposition

If Ψ is a bounded symplectomorphism which preserves $\mathcal{H}_{\mathbb{R}\,*}$ then

$$\Psi \begin{pmatrix} P \\ Q \end{pmatrix} \in \mathcal{L}.$$

Invertibility

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Proof.

If $Q\mathbf{v} = 0$ then by the positive-definiteness of $\overline{Q}^T Q - \overline{P}^T P$

$$0 \leq \overline{\mathbf{v}}^T \left(\overline{Q}^T Q - \overline{P}^T P \right) \mathbf{v} = -\overline{\mathbf{v}}^T \overline{P}^T P\mathbf{v} = -\|P\mathbf{v}\|^2.$$

Thus $Pv = 0$. This implies that $\overline{\mathbf{v}}^T \left(\overline{Q}^T Q - \overline{P}^T P \right) \mathbf{v} = 0$ so $\mathbf{v} = 0$. Thus Q is injective, or equivalently Q has a left inverse. □

Invertibility of A

Note that $A = M_{++}$, the matrix we needed to show was invertible.

Theorem (S, Staubach)

Let $\phi : S^1 \to S^1$ be a quasisymmetry, with

$$\hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}.$$

Then A is invertible and $Z = B\overline{A}^{-1} \in \mathfrak{S}$.

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Then A is invertible and $Z = B\overline{A}^{-1} \in \mathfrak{S}$.

Note: This theorem was proven originally by Takhtajan and Teo. However their proof uses the conformal welding theorem, so we must provide a new one.

Proof

Proof: Invertibility of A:

$$\begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} B \\ \overline{A} \end{pmatrix} \in \mathcal{L}.$$

So $\overline{A}$ has a left inverse.

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So $\overline{A}$ has a left inverse.
Apply to ϕ^{-1} (also a quasisymmetry)

$$\hat{C}_\phi = \begin{pmatrix} \overline{A}^T & -B^T \\ -\overline{B}^T & A^T \end{pmatrix}.$$

So $\overline{A}^T$ has a left inverse; thus A is a bounded bijection so it is invertible.

Proof continued

Let $Z = B\overline{A}^{-1}$.

Proof continued

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Recall:

$$(B, \overline{A}) \in \mathcal{L} \Rightarrow B\overline{A}^{-1} \in \mathfrak{S}.$$

Definition of Grunsky matrix

Let

$$g(z) = g_{-1}z + g_0 + g_1 z + g_2 z^2 + \cdots .$$

The Grunsky matrix b_{mn} of g is defined by

$$\log \frac{g(z) - g(w)}{z - w} = \sum_{m,n=1}^{\infty} b_{mn} z^m w^n.$$

Grunsky matrix and welding maps

Theorem (Takhtajan and Teo)

Let $f(z) = f_1 z + f_2 z^2 + \cdots \in \mathcal{D}(\mathbb{D})$ *and* $g = g_{-1} z + g_-$ *where* $g_- \in \mathcal{D}(\mathbb{D}^*)$, *and let* $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ *be a quasisymmetry. Assume that* $g \circ \phi = f$ *on* $\mathbb{S}^1$. *Let*

$$\hat{C}_\phi = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \overline{\mathfrak{B}} & \overline{\mathfrak{A}} \end{pmatrix} \quad \text{and} \quad \hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \tag{1}$$

1. *If* $g_{-1} \neq 0$, *then the Grunsky matrix of* g *is* $\overline{B} A^{-1}$.
2. *If* $f_1 \neq 0$, *then the Grunsky matrix of* f *is* $\overline{\mathfrak{B}} \mathfrak{A}^{-1}$.

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1. *If* $g_{-1} \neq 0$, *then the Grunsky matrix of* g *is* $\overline{B} A^{-1}$.
2. *If* $f_1 \neq 0$, *then the Grunsky matrix of* f *is* $\overline{\mathfrak{B}} \mathfrak{A}^{-1}$.

Note: Their statement assumes that *f* and *g* are the maps in the conformal welding theorem. However their *proof* only uses the assumptions above and invertibility of *g*.

Recap: proof of conformal welding theorem

Proof:
(1) For a quasisymmetry ϕ.

$$\hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}$$

A is invertible.

Recap: proof of conformal welding theorem

Proof:
(1) For a quasisymmetry ϕ.

$$\hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}$$

A is invertible.
(2) We may find $f, g \in \mathcal{H}$ such that $f \circ \phi^{-1} = g$ using

$$C_{\phi^{-1}} f = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}$$

where $g_+ = g_{-1} z = \alpha z$ and $g_- = g_0 + g_1/z + g_2/z^2 + \cdots$ which has the solution

$$f = M_{++}^{-1} g_+ \quad g_- = M_{+-} f.$$

Note that $M_{++} = A$.

Proof continued

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(4) $Z = B\overline{A}^{-1}$ satsifies $I - Z\overline{Z}$ is positive definite since $Z \in \mathfrak{S}$. Thus $\|Z\| \leq k < 1$ some k.

Proof continued

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(4) $Z = B\overline{A}^{-1}$ satsifies $I - Z\overline{Z}$ is positive definite since $Z \in \mathfrak{S}$. Thus $\|Z\| \leq k < 1$ some k.

(5) By a classical theorem of Pommerenke if $\|Z\| \leq k < 1$ then g is univalent and quasiconformally extendible. A bit of work shows the same for f.

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