A functional-analytic proof of the conformal welding theorem

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Conformal welding theorem

Definition

A quasiconformal map \( \phi : A \to B \) between open connected domains \( A \) and \( B \) in \( \mathbb{C} \) is a homeomorphism such that

1. \( \phi \) is absolutely continuous on almost every vertical and horizontal line in every closed rectangle \([a, b] \times [c, d] \subseteq A\)

2. \( \left\| \frac{\partial f}{\partial x} \right\|_{\infty} \leq k \)

for some fixed \( k < 1 \).
Conformal welding theorem

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2. \[
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\]

for some fixed \( k < 1 \).

Definition

A quasisymmetric map \( \phi : S^1 \to S^1 \) is a homeomorphism which is the boundary values of some quasiconformal map \( H : \mathbb{D} \to \mathbb{D} \).
Conformal welding theorem

Theorem (Conformal welding theorem)

Let $\phi : S^1 \rightarrow S^1$ be a quasisymmetric map and let $\alpha > 0$. There is a pair of maps $f : \mathbb{D} \rightarrow \mathbb{C}$ and $g : \mathbb{D}^* \rightarrow \overline{\mathbb{C}}$ such that

1. $f$ is one-to-one and holomorphic, and has a quasiconformal extension to $\overline{\mathbb{C}}$
2. $g$ is one-to-one and holomorphic except for a simple pole at $\infty$, and has a quasiconformal extension to $\overline{\mathbb{C}}$
3. $f(0) = 0$, $g(\infty) = \infty$ and $g'(\infty) = \alpha$.
4. $\phi = g^{-1} \circ f$ on $S^1$. 
Conformal welding theorem

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- standard proof uses existence and uniqueness to solutions of the Beltrami equation.
- We will give another proof using symplectic geometry and Grunsky inequalities.
The function spaces $\mathcal{H}$ and $\mathcal{H}_*$

Let $\mathcal{H}$ denote the space of $L^2$ functions $h$ on $S^1$ such that

$$\sum_{n=-\infty}^{\infty} |n| |\hat{h}(n)|^2 < \infty.$$ 

Define

$$\|h\|^2 = |\hat{h}(0)|^2 + \sum_{n=-\infty}^{\infty} |n| |\hat{h}(n)|^2.$$
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We will also consider

$$\mathcal{H}_* = \{ h \in \mathcal{H} : \hat{h}(0) = 0 \}$$

with norm

$$\|h\|_*^2 = \sum_{n=-\infty}^{\infty} |n| |\hat{h}(n)|^2.$$
Decomposition of $\mathcal{H}_*$

\[ \mathcal{H}_+ = \{ h \in \mathcal{H}_* : h = \sum_{n=1}^{\infty} h_n e^{i n \theta} \} \]

\[ \mathcal{H}_- = \{ h \in \mathcal{H}_* : h = \sum_{n=-\infty}^{-1} h_n e^{i n \theta} \}. \]
Decomposition of $H_*$

$$H_+ = \{ h \in H_* : h = \sum_{n=1}^{\infty} h_n e^{in\theta} \}$$

$$H_- = \{ h \in H_* : h = \sum_{n=-\infty}^{-1} h_n e^{in\theta} \}.$$ 

It is well-known that we have the following isometries

$$H_+ \cong D(D) = \{ h : D \to \mathbb{C} : \int\int_{D} |h'|^2 \, dA < \infty \, \, h(0) = 0 \}$$

$$H_- \cong D(D^*) = \{ h : D^* \to \mathbb{C} : \int\int_{D^*} |h'|^2 \, dA < \infty \, \, h(\infty) = 0 \}$$

Summarized in Nag and Sullivan.
Composition operators on $H$ and $H^*$

We consider two composition operators

$$C_\phi : H \to H \quad C_\phi h = h \circ \phi$$

$$\hat{C}_\phi : H^* \to H^* \quad \hat{C}_\phi h = h \circ \phi - \frac{1}{2\pi} \int_{S^1} h \circ \phi(e^{i\theta}) \, d\theta.$$
The function spaces and composition operator

Composition operators on $H$ and $H_*$

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Theorem (Nag and Sullivan, quoting notes of Zinsmeister)

$\hat{C}_\phi$ is bounded if $\phi$ is a quasisymmetry.

Theorem (S and Staubach)

If $\phi$ is a quasisymmetry then $C_\phi$ is bounded.
Sketch of a new proof

Treat $\phi$ as a composition operator $C_\phi$ on $\mathcal{H}$: we want to solve for unknown functions $f$ and $g$ in equation $f \circ \phi^{-1} = g, g_{-1} = \alpha$. 

Using the decomposition $\mathcal{H} = [\mathcal{H} + ] \oplus [\mathcal{C} \oplus \mathcal{H} - ]$, the welding equation can be written $C_\phi f = (M^{++} + M^{+} - M^{--} - M^{-}) f_0 = (g^+ + g^-)$ so $M^{++} f = g^+$ and $M^{+-} f = g^-$. 

where $g^+ = g_{-1} z = \alpha z$ and $g^- = g_0 + g_1 / z + g_2 / z^2 + \cdots$. 

which leads to the solution $f = M_{-1}^{+-} g^+ + g^- = M_{-1}^{-+} f$. 

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$$C_\phi f = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}$$

so

$$M_{++} + g_+ = g_{++} = g_{-1} = \alpha$$

and

$$M_{-+} + g_- = g_{-+} = g_{0} + g_1/z + g_2/z^2 + \cdots.$$
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so

$$M_{++}f = g_+ \quad \text{and} \quad M_{+-}f = g_-.$$ 

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which leads to the solution

$$f = M_{++}^{-1} g_+ \quad g_- = M_{+-} f.$$
What are the gaps?

We need to show that

- $M_{++}$ is invertible: will use symplectic geometry and results of Nag and Sullivan, Takhtajan and Teo.
- The solutions in $\mathcal{H}$ so obtained have the desired properties: conformal with quasiconformal extensions: will use Grunsky inequalities.
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Here we go!
Symplectic structure on $\mathcal{H}_*$

For $f, g \in \mathcal{H}_*$ let

$$\omega(f, g) = -i \sum_{n=-\infty}^{\infty} f_n g_{-n}.$$ 

If one restricts to the real subspace (such that $\hat{f}(-n) = \overline{\hat{f}(n)}$) this is a non-degenerate anti-symmetric form $2\text{Im} \left( \sum_{n=1}^{\infty} \hat{f}(n) \hat{g}(n) \right)$. 
The proof
Symplectic geometry of $\mathcal{H}_*$

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Theorem (Nag and Sullivan)

If $\phi : S^1 \to S^1$ is quasisymmetric then $\hat{C}_\phi$ is a symplectomorphism (that is, $\omega(\hat{C}_\phi f, \hat{C}_\phi g) = \omega(f, g)$).

Note that $\hat{C}_\phi$ has the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$
The infinite Siegel disc (Nag and Sullivan)

Definition

The infinite Siegel disc $\mathcal{S}$ is the set of maps $Z : \mathcal{H}_- \rightarrow \mathcal{H}_+$ such that $Z^T = Z$ and $I - ZZ^T$ is positive definite.
The infinite Siegel disc (Nag and Sullivan)

Definition
The infinite Siegel disc $\mathcal{S}$ is the set of maps $Z : H_- \rightarrow H_+$ such that $Z^T = Z$ and $I - ZZ^T$ is positive definite.

Context:
- the graph of each $Z$ is a Lagrangian subspaces of $H_*$
- symplectomorphisms $\hat{C}_\phi$ act on them.

Definition
Let $\mathcal{L}$ be the set of bounded linear maps of the form

$$(P, Q) : H_- \rightarrow H_*$$

where $P : H_- \rightarrow H_+$ and $Q : H_- \rightarrow H_-$ are bounded operators satisfying $\overline{P}^T P - \overline{Q}^T Q > 0$ and $Q^T P = P^T Q$. 
Two facts

• $Q$ invertible $\Rightarrow PQ^{-1} \in \mathcal{G} \iff (P, Q) \in \mathcal{L}$.

• $(P, Q)Q^{-1} = (PQ^{-1}, I)$ has the same image as $(P, Q)$
Invariance of $\mathcal{L}$

$\mathcal{L}$ is invariant under bounded symplectomorphisms.

**Proposition**

*If $\psi$ is a bounded symplectomorphism which preserves $\mathcal{H}_{\mathbb{R}^*}$ then*

$$\psi \left( \begin{array}{c} P \\ Q \end{array} \right) \in \mathcal{L}.$$
Invertibility

Proposition

*If* \((P, Q) \in \mathcal{L}\) *then* \(Q\) *has a left inverse.*
Invertibility

Proposition

If \((P, Q) \in \mathcal{L}\) then \(Q\) has a left inverse.

Proof.

If \(Qv = 0\) then by the positive-definiteness of \(Q^TQ - P^TP\)

\[
0 \leq v^T \left( Q^TQ - P^TP \right) v = -v^TP^TPv = -\|Pv\|^2.
\]

Thus \(Pv = 0\). This implies that \(v^T \left( Q^TQ - P^TP \right) v = 0\) so \(v = 0\). Thus \(Q\) is injective, or equivalently \(Q\) has a left inverse.
Invertibility of $A$

Note that $A = M_{++}$, the matrix we needed to show was invertible.

**Theorem (S, Staubach)**

*Let $\phi : S^1 \to S^1$ be a quasisymmetry, with*

$$\hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ B & \bar{A} \end{pmatrix}.$$

*Then $A$ is invertible and $Z = B\bar{A}^{-1} \in \mathcal{G}$.***
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**Theorem (S, Staubach)**

Let $\phi : S^1 \to S^1$ be a quasisymmetry, with

$$\hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ B & \bar{A} \end{pmatrix}.$$

Then $A$ is invertible and $Z = B\bar{A}^{-1} \in \mathcal{S}$.

**Note:** This theorem was proven originally by Takhtajan and Teo. However their proof uses the conformal welding theorem, so we must provide a new one.
Proof

Proof: Invertibility of $A$:

\[
\begin{pmatrix} A & B \\ B & A \end{pmatrix} \cdot \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} B \\ A \end{pmatrix} \in \mathcal{L}.
\]

So $\bar{A}$ has a left inverse.
Proof: Invertibility of $A$:

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\begin{pmatrix}
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B & A
\end{pmatrix} \cdot \begin{pmatrix}
0 \\
I
\end{pmatrix} = \begin{pmatrix}
B \\
A
\end{pmatrix} \in \mathcal{L}.
\]

So $A$ has a left inverse.

Apply to $\phi^{-1}$ (also a quasisymmetry)

\[
\hat{C}_\phi = \begin{pmatrix}
A^T & -B^T \\
-B^T & A^T
\end{pmatrix}.
\]

So $A^T$ has a left inverse; thus $A$ is a bounded bijection so it is invertible.
Let $Z = B \overline{A}^{-1}$. 
Proof continued

Let $Z = B\bar{A}^{-1}$.

Recall:

$$(B, \bar{A}) \in \mathcal{L} \Rightarrow B\bar{A}^{-1} \in \mathcal{G}.$$
Definition of Grunsky matrix

Let

\[ g(z) = g_{-1}z + g_0 + g_1z + g_2z^2 + \cdots. \]

The Grunsky matrix \( b_{mn} \) of \( g \) is defined by

\[
\log \frac{g(z) - g(w)}{z - w} = \sum_{m,n=1}^{\infty} b_{mn}z^m w^n.
\]
Grunsky matrix and welding maps

Theorem (Takhtajan and Teo)

Let \( f(z) = f_1 z + f_2 z^2 + \cdots \in \mathcal{D}(\mathbb{D}) \) and \( g = g_- z + g_- \) where \( g_- \in \mathcal{D}(\mathbb{D}^*) \), and let \( \phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1 \) be a quasisymmetry. Assume that \( g \circ \phi = f \) on \( \mathbb{S}^1 \). Let

\[
\hat{C}_\phi = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \quad \text{and} \quad \hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \quad (1)
\]

1. If \( g_- \neq 0 \), then the Grunsky matrix of \( g \) is \( \overline{BA}^{-1} \).
2. If \( f_1 \neq 0 \), then the Grunsky matrix of \( f \) is \( \overline{BA}^{-1} \).
Grunsky matrix and welding maps

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\[
\hat{C}_\phi = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \quad \text{and} \quad \hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}
\]  \hspace{1cm} (1)

1. If \( g_{-1} \neq 0 \), then the Grunsky matrix of \( g \) is \( \overline{B}A^{-1} \).
2. If \( f_1 \neq 0 \), then the Grunsky matrix of \( f \) is \( \overline{B}A^{-1} \).

Note: Their statement assumes that \( f \) and \( g \) are the maps in the conformal welding theorem. However, their proof only uses the assumptions above and invertibility of \( g \).
Recap: proof of conformal welding theorem

Proof:
(1) For a quasisymmetry \( \phi \).

\[
\hat{C}_{\phi^{-1}} = \begin{pmatrix}
A & B \\
\bar{B} & \bar{A}
\end{pmatrix}
\]

A is invertible.
Recap: proof of conformal welding theorem

**Proof:**

1. For a quasisymmetry $\phi$.

   $$\hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

   $A$ is invertible.

2. We may find $f, g \in \mathcal{H}$ such that $f \circ \phi^{-1} = g$ using

   $$C_{\phi^{-1}} f = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}$$

   where $g_+ = g_- z = \alpha z$ and $g_- = g_0 + g_1/z + g_2/z^2 + \cdots$ which has the solution

   $$f = M_{++}^{-1} g_+ \quad g_- = M_{--} f.$$

   Note that $M_{++} = A$.  

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Proof continued

(3) $BA^{-1}$ is the Grunsky matrix of $g$ under these assumptions, by the theorem of Takhtajan and Teo.
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(4) $Z = BA^{-1}$ satsifies $I - ZZ$ is positive definite since $Z \in \mathcal{S}$. Thus $\|Z\| \leq k < 1$ some $k$. 

Proof continued

(3) $B\bar{A}^{-1}$ is the Grunsky matrix of $g$ under these assumptions, by the theorem of Takhtajan and Teo.

(4) $Z = B\bar{A}^{-1}$ satisfies $I - ZZ$ is positive definite since $Z \in \mathbb{S}$. Thus $\|Z\| \leq k < 1$ some $k$.

(5) By a classical theorem of Pommerenke if $\|Z\| \leq k < 1$ then $g$ is univalent and quasiconformally extendible. A bit of work shows the same for $f$. 
References


