Some analytic problems in conformal field theory

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Goals etc

Work is joint with either David Radnell (American Univ. of Sharjah). or Wolfgang Staubach (Uppsala Universitet) or both.

Goals - Rigorously construct two-dimensional conformal field theory according to Segal, Kontsevich and others.

- Resolve analytic issues arising in a program of Yi-Zhi Huang (Rutgers) for constructing CFT from vertex operator algebras

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- Partly we are exploiting the overlap between the fields to get new results in geometric function theory and Teichmüller theory.

- My own goal: certain approach to conformal invariants and index theorems.

Teichmüller theory

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Deformations of Riemann surfaces

Quasiconformal Teichmüller theory in a nutshell:

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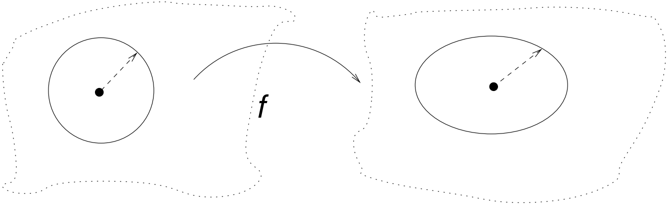
1. Teichmüller space is the set of deformations of Riemann surfaces
2. Local deformations are quasiconformal maps.
3. It imposes a *weaker* equivalence than conformal equivalence.
4. Teichmüller space can be modelled by function spaces (in several ways); locally it is a Banach space.

Idea of quasiconformal map



A quasiconformal map is one such that the Jacobian matrix takes circles to ellipses of bounded distortion: i.e. ratio of the major to minor axes is bounded by a fixed constant.

Idea: angle is distorted.

Precise definition

Let $A, B \subset \mathbb{C}$ be open and connected.

Definition

A **quasiconformal map** $f : A \to B$ is an orientation-preserving homeomorphism such that

1. f is absolutely continuous on horizontal and vertical lines
2. There is a fixed constant $k < 1$ such that

$$\left| \frac{\partial f}{\partial \bar{z}} \right| \leq k \left| \frac{\partial f}{\partial z} \right|$$

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Quasiconformal maps have a *weaker local condition* but a *stronger global condition* than a diffeomorphism.

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$$\{(\Sigma, f, \Sigma_1)\}/\sim$$

where $f : \Sigma \to \Sigma_1$ is quasiconformal and

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rel boundary means the homotopy is the identity map on the boundary of Σ.

Teichmüller equivalence

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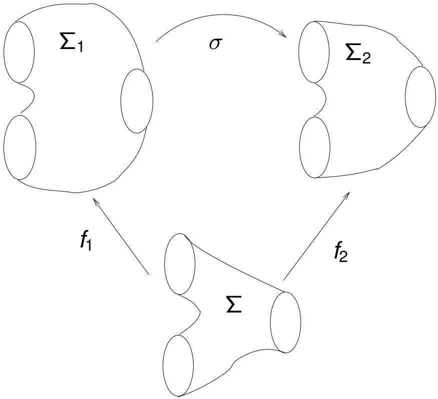

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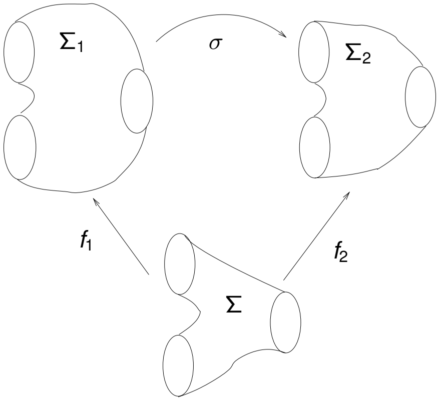

- Σ compact minus points $\Rightarrow T(\Sigma)$ finite dimensional.

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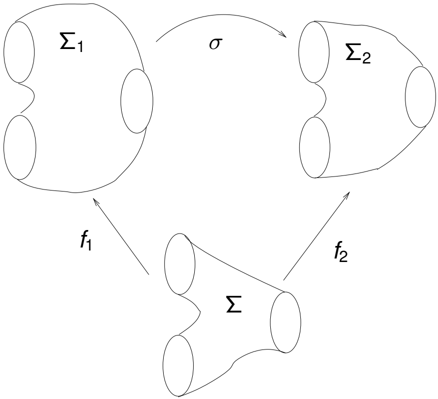

- Σ compact minus points ⇒ $T(\Sigma)$ finite dimensional.
- Σ has boundary curves ⇒ $T(\Sigma)$ infinite dimensional.

Complex structures on Teichmüller space

Theorem (Ahlfors, 1960)

Let R be a compact Riemann surface of genus g with n points removed. Assume that $2g - 2 + n > 0$. The Teichmüller space of R is a $3g - 3 + n$ dimensional complex manifold.

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More generally,

Theorem (Bers, 1964,1965)

The Teichmüller space of a Riemann surface is a complex Banach manifold.

Example: universal Teichmüller space

Let Σ be the disc $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$.
Let $\mu \in L^\infty(\mathbb{D}^*)$, $\|\mu\|_\infty = k < 1$. Let $f_\mu : \mathbb{D}^* \to \Sigma_1$ be the solution to the Beltrami differential equation

$$\frac{\overline{\partial} f}{\partial f} = \mu.$$

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Let g_μ be the quasiconformal map satisfying

1. $\overline{\partial} g / \partial g = \mu$ on $\mathbb{D}^*$
2. $\overline{\partial} g / \partial g = 0$ on $\mathbb{D}$
3. $g(0) = 0$, $g'(0) = 1$, $g''(0) = 0$.

Teichmüller equivalence: $[\mathbb{D}^*, f_\mu, \Sigma_1] = [\mathbb{D}^*, f_\nu, \Sigma_2]$ if and only if $g_\mu = g_\nu$ on $\mathbb{D}$.

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So $T(\mathbb{D}^*)$ is a function space of conformal maps $g : \mathbb{D} \to \mathbb{C}$.

Example continued

Let $S(g)$ denote the Schwarzian derivative

$$S(g) = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right).$$

Theorem (Bers, classical)

The set of Schwarzian derivatives of g arising from the Teichmüller space above is an open subset of the Banach space of holomorphic functions

$$\{h : \mathbb{D} \to \mathbb{C} \ : \ \|(1-|z|^2)^2 h(z)\|_\infty < \infty\}.$$

Quasisymmetries

Quasisymmetries $= \mathrm{QS}(\mathbb{S}^1) =$ boundary values of quasiconformal maps of $\mathbb{D}$.

$$\mathrm{AnalyticDiff}(\mathbb{S}^1) \subsetneq \mathrm{Diff}(\mathbb{S}^1) \subsetneq \mathrm{QS}(\mathbb{S}^1) \subsetneq \mathrm{Homeo}(\mathbb{S}^1).$$

The universal Teichmüller space is in natural one-to-one correspondence with

$$T(\mathbb{D}^*) \cong \mathrm{QS}(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1).$$

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Relation to CFT/representations of *Diff*$(\mathbb{S}^1)$/string theory in various forms recognized by Bowick and Rajeev, Nag and Verjovsky, Kirillov, Neretin, Nag and Sullivan, etc, etc.

Conformal field theory

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What is conformal field theory?

Conformal Field Theory (CFT) is:

- Special class of quantum/statistical field theories, invariant under local rescaling and rotation.
- Mathematical definition (G. Segal, Kontsevich $\approx$ 1986).
- Requires results in algebra, topology and analysis.
- Related to vertex operator algebras, representations of infinite-dimensional Lie algebras.

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Our General Aim:

- Provide a natural analytic setting for the rigorous definition of CFT in higher genus.
- Use CFT ideas to prove new results in Teichmüller theory and geometric function theory.

Rigged moduli space of Friedan and Shenker/Vafa

Let Σ be a bordered Riemann surface of genus g with n boundary curves $\partial_i \Sigma$ homeomorphic to $\mathbb{S}^1$.

Definition (Riggings)

A "rigging" is an n-tuple of maps $\phi = (\phi_1, \ldots, \phi_n)$ where $\phi_i : \mathbb{S}^1 \to \partial_i \Sigma$ is a parametrization of the boundary.

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Equivalence relation: $(\Sigma, \phi) \sim (\Xi, \psi)$ if and only if there is a biholomorphism $\sigma : \Sigma \to \Xi$ such that $\psi_i = \sigma \circ \phi_i$ for all i.

Definition (Rigged moduli space)

The rigged moduli space of bordered Riemann surfaces of genus g with n boundary curves is

$$\widetilde{\mathcal{M}}(g, n) = \{(\Sigma, \phi)\}/\sim .$$

Riggings: important analytic point

Question sometimes ignored: how regular are the riggings?
Must include at the very least all analytic diffeomorphisms
$\phi_i : \mathbb{S}^1 \to \partial_i \Sigma$.

The question of what choice of riggings is fundamental.

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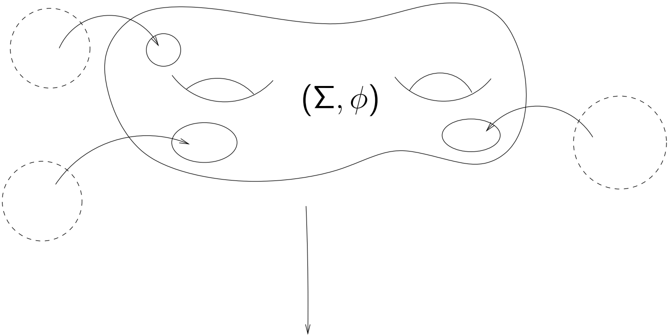
The question of what choice of riggings is fundamental.

Radnell, Staubach and I assume that each map ϕ_i is a quasisymmetry.
However that may be too big a class.

Needs of conformal field theory I

Let $\mathcal{H}$ be a Hilbert space.



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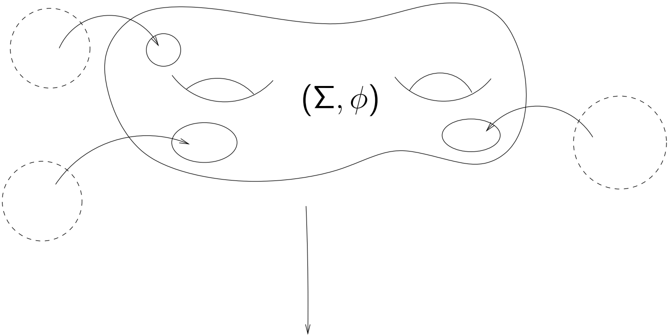

$$A(\Sigma, \phi) : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$$

It is required that $A(\Sigma, \phi)$ depends holomorphically on (Σ, ϕ).

Needs of conformal field theory II

- So need a complex structure on the rigged moduli space $\widetilde{\mathcal{M}}(g,n)$.

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- There is also an algebraic structure: “sewing” Riemann surfaces. This should also be holomorphic.

Results Part I

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Rigged moduli space is almost Teichmüller space

Theorem (Radnell and S, 2006 Commun. Contemp. Math.)

Let Σ be a Riemann surface of genus g bordered by n closed curves.

$$\widetilde{\mathcal{M}}(g,n) = T(\Sigma)/G.$$

The action by G is fixed-point-free and properly discontinuous.

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Consequences for conformal field theory

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The sewing operation on rigged moduli space is holomorphic.

Some consequences for Teichmüller theory

1. Teichmüller space of a bordered surface is holomorphically fibered over the Teichmüller space of a punctured surface obtained by sewing on discs. [Radnell & S, Journal d'Analyse 2009, Conf. Geom. and Dynamics 2010]

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2. New coordinates on the infinite-dimensional Teichmüller space (same refs as above)
3. For a doubly-connected Riemann surface A, $T(A)/\mathbb{Z}$ is a semigroup with holomorphic multiplication (Neretin-Segal semigroup). [Radnell and S, J. Lond. Math. Soc 2012]

WP-class Teichmüller space *or* What kind of riggings?

Weil-Petersson class Teichmüller space

or

What kind of riggings?

WP-class Teichmüller space

The universal Teichmüller space is "too big".
Hui, Cui, Takhtajan and Teo:
Model I:

Definition

The WP-class universal Teichmüller space is the subset $T_{WP}(\mathbb{D}^*)$ of $T(\mathbb{D}^*)$ whose elements are represented by conformal maps $g : \mathbb{D} \to \mathbb{C}$ such that

$$\iint_{\mathbb{D}} \left| \frac{g''}{g'} \right|^2 dA < \infty.$$

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Model II

Definition

The WP-class universal Teichmüller space is the subset $T_{WP}(\mathbb{D}^*)$ of $T(\mathbb{D}^*)$ whose elements are represented by elements $\phi \in \mathrm{QS}(\mathbb{S}^1)$ such that ϕ is absolutely continuous and $\log|\phi'| \in H^{1/2}$ [Shen, 2013].

Monograph of Takhtajan and Teo

Revolutionary monograph of Takhtajan and Teo:

1. $T_{WP}(\mathbb{D}^*)$ is a Hilbert manifold.
2. $T_{WP}(\mathbb{D}^*)$ is a topological group.
3. The Weil-Petersson metric converges, and is Kähler-Einstein.
4. $T_{WP}(\mathbb{D}^*)$ embeds holomorphically in the Segal-Wilson universal Grassmanian
5. Kähler potentials for WP metric, etc. etc. etc.

Results Part II

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Extension to arbitrary bordered surfaces

Theorem (Radnell, S, and Staubach 2012, submitted)

A bordered Riemann surface Σ of genus g with n boundary curves homeomorphic to $\mathbb{S}^1$ possesses a Teichmüller space $T_{WP}(\Sigma)$ with a Hilbert manifold structure. The inclusion $T_{WP}(\Sigma) \hookrightarrow T(\Sigma)$ is holomorphic.

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Proof.

Many challenging analysis problems. □

Needs of CFT III: Families of Cauchy-Riemann operators

• The parametrization $\phi : \mathbb{S}^1 \to \partial_i \Sigma$ induces a decomposition of Fourier series of functions on the boundary
$C^i_{\pm} = \{f : \partial_i\Sigma \to \mathbb{C} \; : \; f \circ \phi_i \;\text{ has only }\; \pm \text{ Fourier coefficients}\}.$

$$\pi : Hol(\Sigma) \to C^1_+ \oplus \cdots \oplus C^n_+$$
$$f \mapsto \left(P^1_+ \, f|_{\partial_1 \Sigma}, \ldots, P^1_+ \, f|_{\partial_1 \Sigma} \right).$$

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• π depends both on Σ and the rigging.

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2. Need eventually to establish certain sewing relations of the determinant line bundle of this operator
3. Existence of the determinant line requires sufficient regularity, which is closely tied to two questions:
    1. What is the regularity of the riggings?
    2. What further regularity do you impose on $Hol(\Sigma)$ (i.e. at the boundary)?

Remark: this is closely related to convergence of the Weil-Petersson metric.

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Problem of construction of the determinant line of π reduces to the Plemelj-Sokhotski jump formula on n boundary curves (Yi-Zhi Huang). (in genus zero; higher genus introduces the interesting *algebraic* problems).

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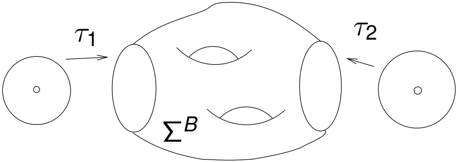

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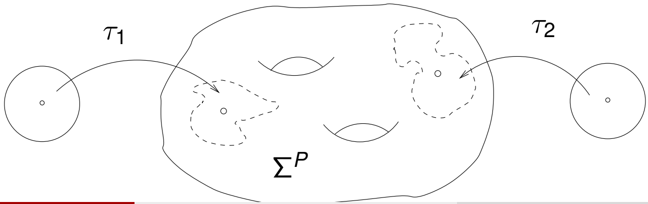

Solution to analytic problems

Theorem (Radnell, S, Staubach 2013, submitted)

(genus zero case) Let $\Sigma \subset \mathbb{C}_\infty$ be an open connected set bordered by n WP-class Jordan curves.

- *Complex harmonic functions on Σ of finite Dirichlet energy are precisely those with boundary values in a certain Besov space $\mathcal{H}$ on $\partial_i \Sigma$*
- *Extension and restriction operators are bounded.*
- *The Plemelj-Sokhotski jump decomposition is bounded on this Besov space.*

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- *Extension and restriction operators are bounded.*
- *The Plemelj-Sokhotski jump decomposition is bounded on this Besov space.*

Remark 1: all analytic problems already appear in genus zero.
Remark 2: all function spaces appearing are conformally invariant.
Remark 3: Takhtajan/Teo technology can be used to show the determinant exists.

What else

Analytically we've cleared a path, but there are some issues left:

1. Loewner theory in WP-class Teichmüller space (not necessarily randomized).
2. Prove holomorphicity of sewing for WP-class Teichmüller space/rigged moduli space

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But, we are now (finally!) in a position to make rigorous **geometric** and **algebraic** constructions.

1. Determinant line bundle (genus zero easy)
2. Sewing properties of determinant line bundle.
3. embeddings of Teichmüller space into the infinite Grassmanian in genus g and n boundary curves.
4. Connect CFT determinant line bundle to local index theorems.
5. Curvature of WP metric.

The end

Thanks!