Areas and Distances

MATH 1700

Readings

Section 5.1

The problem

Problem: How do you find the area of a shape whose sides are not straight?

Right and left sums

Partial answer: We can approximate the area using rectangles. Let $f(x)$ be a function defined on the interval $[a, b]$. For an integer n set $\Delta x = (b - a)/n$ and let $x_i = a + i\Delta x$.

Definition

The **right sum** of f over $[a, b]$ corresponding to the points $\{x_0, \ldots, x_n\}$ is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + \cdots f(x_n)\Delta x$$

The **left sum** of f over $[a, b]$ corresponding to $\{x_0, \ldots, x_n\}$ is

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = f(x_0)\Delta x + \cdots f(x_{n-1})\Delta x.$$

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Idea: if the number of points n is large, this is close to the actual area.

General formula

The points x_i^* could be in the middle instead: $x_{i-1} \leq x_i^* \leq x_i$, and you still get an approximation to the area.

Definition

The **Riemann sum** associated with the points $x_1^*, \ldots, x_n^*$ is

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Definition of net area

Definition (working definition)

The **net area** under the graph of $y = f(x)$ between $x = a$ and $x = b$ is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Note: area can be negative with this definition!

A motivating example: distance

If $v = f(t)$ describes the velocity v as a function of time t (in a straight line!)

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If $v = f(t)$ describes the velocity v as a function of time t (in a straight line!)

then the distance travelled between $t = a$ and $t = b$ is the area under the graph between a and b.