[6] 1. Evaluate $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}$.

$\underbrace{\left(1+\frac{2}{x}\right)}_{\to 1}{}^{\overbrace{3x}^{\to\infty}}$ indeterminate

$$\lim_{x\to\infty} \underset{(A)}{3x \ln\left(1+\tfrac{2}{x}\right)} = \lim_{x\to\infty} \frac{3\ln(1+2/x) \to 0}{1/x \to 0}$$ try L'Hôpital's rule

$$\overset{?}{=} \lim_{x\to\infty} \frac{-\frac{6}{x^2}\cdot\frac{1}{1+2/x}}{-1/x^2} = \lim_{x\to\infty}\frac{6}{1+\frac{2}{x}} = 6$$

Since this exists, the limit (A) is 6.

$$\text{So } \lim_{x\to\infty}\left(1+\tfrac{2}{x}\right)^{3x} = \lim_{x\to\infty} e^{3x\ln(1+2/x)}$$
$$= e^6$$

[5] 2. Let $f(x) = \int_{\exp(x^2)}^{5} (3\sin(t^2)+4)dt$. Find the derivative $f'(x)$.

$$f'(x) = \frac{d}{dx}\left(-\int_5^{\exp(x^2)} (3\sin(t^2)+4)dt\right)$$

FTC I and chain rule

$$= -2x e^{x^2}\left(3\sin\left(\left(e^{x^2}\right)^2\right)+4\right)$$

3. Evaluate the following definite integrals.

[6] (a) $\displaystyle\int_1^{\exp(\pi/4)} \frac{\sin(\ln x)}{x}\,dx.$

$u = \ln x$

$du = \frac{1}{x}dx$

$$= \int_0^{\pi/4} \sin u \, du$$

FTC II

$$= \Big|_0^{\pi/4} -\cos u = -\cos\left(\frac{\pi}{4}\right) - (-\cos 0)$$

$$= 1 - \frac{1}{\sqrt{2}}$$

[7] (b) $\displaystyle\int_0^{\frac{\pi}{4}} x\cos(2x)dx.$

$u = x \qquad du = dx$

$dv = \cos(2x) \qquad v = \frac{\sin(2x)}{2}$

$$= \Big|_0^{\pi/4} \frac{x}{2}\sin(2x) - \int_0^{\pi/4} \frac{\sin(2x)}{2}dx$$

$$= \Big|_0^{\pi/4} \frac{x}{2}\sin(2x) + \Big|_0^{\pi/4} \frac{\cos(2x)}{4}$$

$$= \frac{\pi}{4}\sin\left(2\cdot\frac{\pi}{4}\right) - 0 + \frac{\cos\left(\frac{2\pi}{4}\right)}{4} - \frac{\cos(0)}{4}$$

$$= \frac{\pi}{8} - 0 + 0 - \frac{1}{4} = \frac{\pi}{8} - \frac{1}{4}$$

[12] 4. Find the area of the region bounded by $y = 3 - x$ and $y = \dfrac{2}{x}$.

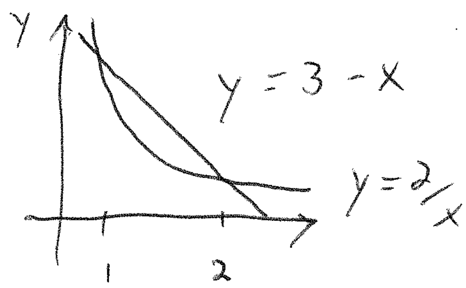
Find the points of intersection:

$$3 - x = \frac{2}{x} \iff 3x - x^2 = 2 \iff x^2 - 3x + 2 = 0$$

$$\iff (x-2)(x-1) = 0$$

$$\iff x = 1 \text{ or } x = 2.$$

which is greater?



$3 - x > \frac{2}{x}$ on interval

$$\text{Area} = \int_1^2 \left[(3 - x) - \frac{2}{x}\right] dx$$

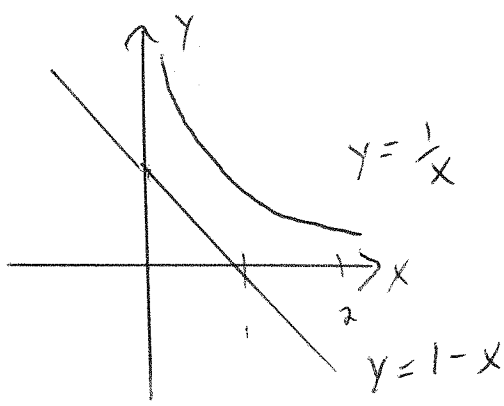
$$= \Big|_1^2 \; 3x - \frac{x^2}{2} - 2\ln|x|$$

$$= 6 - \frac{2^2}{2} - 2\ln 2 - 3 + \frac{1}{2} - 2\ln|1|$$

$$= \frac{3}{2} - 2\ln 2$$

5. Let $\mathcal{T}$ be the region of the plane bounded by $y = \frac{1}{x}$, $y = 1 - x$, $x = 1$, and $x = 2$.

[5] (a) Write an integral that represents the volume of the solid obtained by rotating $\mathcal{T}$ around the line $y = -3$. Use the washer method. **DO NOT EVALUATE** the integral.



if $x \geq 1$ then $1 - x \leq 0 < \frac{1}{x}$

so $r_{out} = \left(\frac{1}{x} - (-3)\right) = \frac{1}{x} + 3$

$r_{in} = \left(1 - x - (-3)\right) = 4 - x$

So $\text{Volume} = \int_1^2 \pi\left(r_{out}^2 - r_{in}^2\right) dx$

$= \int_1^2 \pi\left(\left(\frac{1}{x} + 3\right)^2 - (4 - x)^2\right) dx$

[5] (b) Write an integral that represents the volume of the solid obtained by rotating $\mathcal{T}$ around the y-axis. Use the cylindrical shell method. **DO NOT EVALUATE** the integral.

$\text{Volume} = \int_1^2 2\pi x \left[\frac{1}{x} - (1 - x)\right] dx$

$1 - x \leq 0 < \frac{1}{x}$ on $x \geq 1$

6. Find the following indefinite integrals.

[7] (a) $\int \tan^{62}(x)\sec^4(x)dx.$

$\sec^2 x = 1 + \tan^2 x$

$$= \int \tan^{62} x \,(1+\tan^2 x)\, \sec^2 x \, dx$$

$$= \int \left[\tan^{62} x + \tan^{64} x\right] \cdot \sec^2 x \, dx$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$= \int (u^{62} + u^{64})\, du$$

$$= \frac{u^{63}}{63} + \frac{u^{65}}{65} + C$$

$$= \frac{\tan^{63} x}{63} + \frac{\tan^{65} x}{65} + C$$

[7] (b) $\int \frac{1}{x^2\sqrt{9-x^2}}dx.$

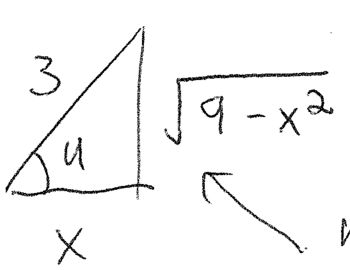
$x = 3\sin u$ $\quad -\frac{\pi}{2} < u < \frac{\pi}{2}$
$dx = 3\cos u \, du$

$$= \int \frac{3\cos u \, du}{9\sin^2 u \sqrt{9 - 9\sin^2 u}} = \int \frac{3\cos u \, du}{9\sin^2 u \cdot \sqrt{9\cos^2 u}}$$

for $-\frac{\pi}{2} < u < \frac{\pi}{2}$, $\cos u > 0$ so $\sqrt{\cos^2 u} = |\cos u| = \cos u.$

$$= \frac{1}{9}\int \frac{1}{\sin^2 u} du = \frac{1}{9}\int \csc^2 u \, du$$

$$= -\frac{1}{9}\cot u + C = -\frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C$$



using $\frac{x}{3} = \sin u$

so $\cot u = \frac{x}{\sqrt{9-x^2}}$