UNIVERSITY OF MANITOBA

DATE: December 14, 2006 | FINAL EXAMINATION

PAPER # 406 |

DEPARTMENT & COURSE NO: MATH 1500 | TIME: 2 hours

EXAMINATION: Intro Calculus | EXAMINER: Various

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[25] 1. Find $\frac{dy}{dx}$ for the following (**DO NOT SIMPLIFY**):

(a) $y = e^{5x}\ln(2-x^3)$

$$y' = e^{5x}\cdot 5\cdot \ln(2-x^3) + \frac{-3x^2}{2-x^3}\cdot e^{5x}$$

(b) $y = \frac{x^2}{2^x}$

$$y' = \frac{2x\cdot 2^x - 2^x \ln 2 \cdot x^2}{(2^x)^2}$$

(c) $y = (\ln x)^{(x^2)}$

$$\ln y = \ln\left[(\ln x)^{(x^2)}\right] = \overbrace{x^2 \cdot \ln(\ln x)}^{\text{product rule}}$$

$$\frac{1}{y}\cdot y' = 2x\cdot \ln(\ln x) + \frac{1/x}{\ln x}\cdot x^2$$

$$y' = (\ln x)^{(x^2)}\cdot [\quad]$$

(d) $x^y = y^x$

$$\ln x^y = \ln y^x \Rightarrow y\cdot \ln x = x\cdot \ln y$$

$$y'\ln x + \frac{1}{x}y = \ln y + \frac{1}{y}\cdot y'\cdot x$$

$$y'\left(\ln x - \frac{x}{y}\right) = \ln y - y/x$$

$$y' = \frac{\ln y - y/x}{\ln x - x/y}$$

(tricky!)

[10] 2. Prove the following: If $f'(x) = 0$ for all x in an interval I, then f is constant on I.

Choose 2 numbers x_1 & x_2 in I such that $x_1 < x_2$.

Since $f'(x)=0$ on $I \Rightarrow f$ is differentiable on (x_1, x_2) & continuous on $[x_1, x_2]$.

Thus, the Mean Value Thm' says there exists a c $(x_1 < c < x_2)$ such that

$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$

but $f'(c) = 0 \Rightarrow f(x_2) - f(x_1) = 0$

$\Rightarrow f(x_2) = f(x_1)$ &

thus $f(x)$ is constant on I

8(4)/16

[30] 3. If $f(x) = \dfrac{8(x-2)}{x^2}$ then $f'(x) = \dfrac{-8(x-4)}{x^3}$ and $f''(x) = \dfrac{16(x-6)}{x^4}$.

(a) Find all intercepts of the function; state the domain.

Domain: $x \neq 0$

x int $(y=0)$: $x=2$

y int $(x=0)$: NONE

(b) Calculate all limits associated with any horizontal and vertical asymptotes to the curve $y = f(x)$. Also, give the equations of these asymptotes, if any.

H.A.: $\displaystyle\lim_{x\to\pm\infty} \frac{8x-16}{x^2} = \lim_{x\to\pm\infty} \frac{8x/x^2 - 16/x^2}{x^2/x^2} = \lim_{x\to\pm\infty} \frac{8/x - 16/x^2}{1}$ $\to 0$

so there is an H.A. at $y=0$

V.A.: $\displaystyle\lim_{x\to 0^-} f(x) = -\infty$, $\displaystyle\lim_{x\to 0^+} f(x) = -\infty$

-8 -24 -8 0

-2 -1 0 1 2

(c) Find the critical points of $f(x)$, the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing. Find the coordinates of any local maxima and/or minima of $f(x)$.

C.P.'s at $x=4$ & $x=0$

inc: $(0,4)$

dec: $(-\infty, 0)$ $(4, \infty)$

0 4

$f'(-10) = \dfrac{(-)(-)}{(-)}$ $f'(1) = \dfrac{(-)(-)}{(+)}$ $f'(10) = \dfrac{(-)(+)}{(+)}$

NO LOCAL MIN (NOT IN DOMAIN)

LOCAL MAX at $(4, 1)$

(d) Find where the $f(x)$ is concave up, where $f(x)$ is concave down. Find inflection points if any.

Possible IP's at $x=6$, $x=0$

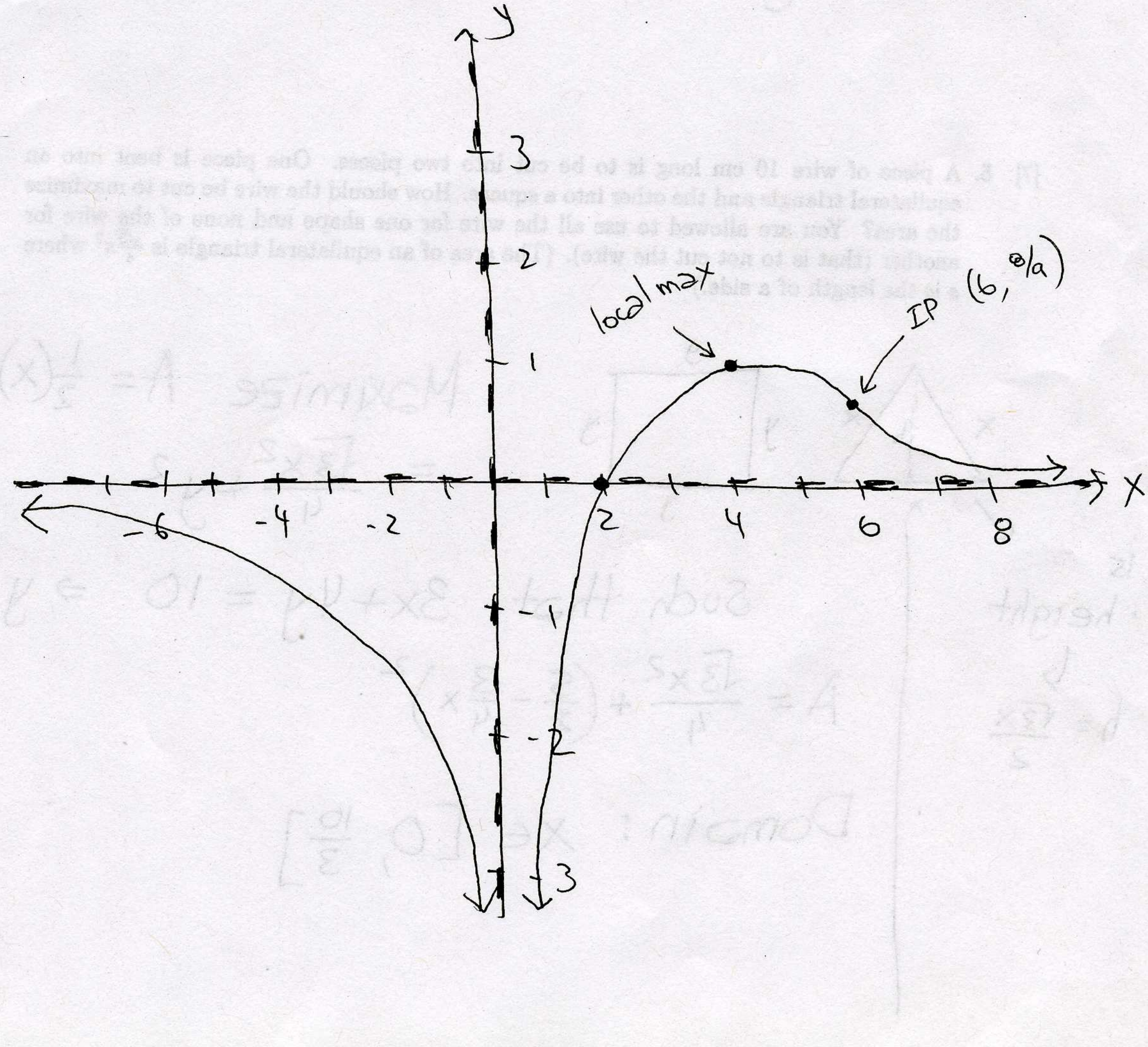
CD CD CU

0 6

$f''(-10) = \frac{-}{+}$ $f''(1) = \frac{-}{+}$ $f''(10) = \frac{+}{+}$

IP at $(6, 8/9)$

(e) Sketch the curve $y = f(x)$, displaying the information found in parts (a),(b), (c) and, (d).

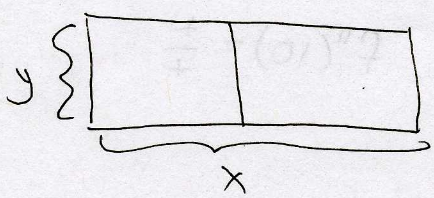


$x^2 = h^2 + \frac{1}{4}x^2$

$\sqrt{\frac{3}{4}x^2} = h^2$

For the following two questions it is sufficient to write down a function and indicate an interval on which it is defined. The function and interval should be chosen so that in order to solve the problem you would find the maximum or minimum value of the function on that interval. **IT IS NOT NECESSARY TO ACTUALLY FIND THE MAXIMUM OR MINIMUM VALUE.**

[7] 4. Two equal adjacent rectangular areas are to be created with fencing. (The two areas are to be separated with one line of fencing.) If 100 m of fence is available, what is the largest total area which can be fenced?



Maximize $A = xy$

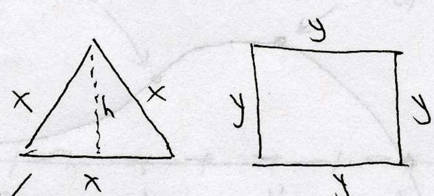
Such that $3y + 2x = 100$ (constraint)

$\Rightarrow x = 50 - \frac{3}{2}y$

$\Rightarrow A = \left(50 - \frac{3}{2}y\right)y$

Domain: $y \in \left[0, \frac{100}{3}\right]$

[7] 5. A piece of wire 10 cm long is to be cut into two pieces. One piece is bent into an equilateral triangle and the other into a square. How should the wire be cut to maximize the area? You are allowed to use all the wire for one shape and none of the wire for another (that is to not cut the wire). (The area of an equilateral triangle is $\frac{\sqrt{3}}{4}s^2$ where s is the length of a side.)



Maximize $A = \frac{1}{2}(x)\left(\frac{\sqrt{3}x}{2}\right) + y^2$

$= \frac{\sqrt{3}x^2}{4} + y^2$

area is base · height

x

$h = \frac{\sqrt{3}x}{2}$

Such that $3x + 4y = 10 \Rightarrow y = \frac{5}{2} - \frac{3}{4}x$

$A = \frac{\sqrt{3}x^2}{4} + \left(\frac{5}{2} - \frac{3}{4}x\right)^2$

Domain: $x \in \left[0, \frac{10}{3}\right]$

$-2+3+12+1$ $\quad 2+3-12+1$

[13] **6. Find the absolute maximum and absolute minimum values of the function $f(x) = 2x^3 + 3x^2 - 12x + 1$ on the interval $[-1, 2]$.**

NOT IN I

$$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2)$$

$$= 6(x+2)(x-1) \Rightarrow \text{C.P.'s } \cancel{x=-2},\ x = 1$$

$f(-1) = 14 \leftarrow$ ABS MAX

$f(1) = -6 \leftarrow$ ABS MN

$f(2) = 16 + 12 - 24 + 1 = 5$

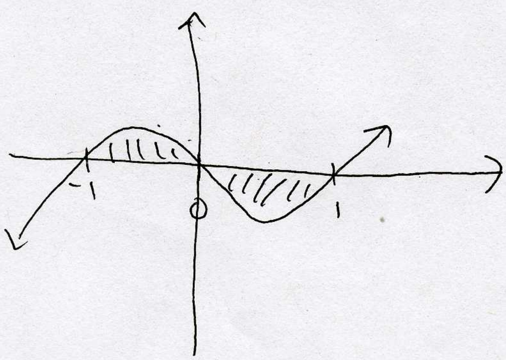
[4] **7. Let $f(x) = \int_0^x (e^{\cos t} - \tan t)\, dt$ for $-\pi/2 < x < \pi/2$. Find $f'(0)$.**

$$f'(x) = e^{\cos x} - \tan x \quad (\text{by FTC \#1})$$

$$\Rightarrow f'(0) = e^{\cos 0} - \tan 0 = e^1 - 0 = e^1$$

$$= (x^2 - x)(x+1) = x^3 + x^2 - x^2 - x$$

[12] **8. Find the total area enclosed by the curve $y = x(x-1)(x+1)$ and the x-axis.**



$$\text{area} = \int_{-1}^{0} (x^3 - x)\,dx + \left(-\int_0^1 (x^3 - x)\,dx\right)$$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} - \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_0^1$$

$$= 0 - \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2} - 0\right)$$

$$= \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{2}{4} = 1/2 \,//$$

[12] 9. Evaluate the following integrals:

(a) $\int (x^4 - \frac{1}{x^2} + e^x + \sec^2(2x))\,dx$

$$= \frac{x^5}{5} - \left(\frac{x^{-1}}{-1}\right) + e^x + \frac{\tan(2x)}{2} + C$$

(b) $\int_{-1}^{1} (x^3 - x)\,dx$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{2} - \left(\frac{1}{4} - \frac{1}{2}\right) = 0$$

(or see question #8!)