THE UNIVERSITY OF MANITOBA

March 11, 2011 TERM TEST 2

DEPARTMENT & COURSE NO: Mathematics 2730

EXAMINATION: Sequences and Series TIME: 1 HOUR EXAMINER: Kalajdzievski

LAST NAME (Family Name): (Print in ink) ______________________________

FIRST NAME (Given Name): (Print in ink) ______________________________

STUDENT NUMBER: ____________________ SEAT NUMBER: ____________

SIGNATURE: (In ink) ______________________________

INSTRUCTIONS TO CANDIDATES:

This is an 1-hour exam. **Please show your work clearly**. Please justify your answers, unless otherwise stated.

No calculators or other aids are permitted.

This exam has a title page, 5 pages of questions and 1 blank page for rough work. Please check that you have all the pages.

The value of each question is indicated in the left-hand margin beside the statement of the question. The total value of all questions is 42 (40 is 100%).

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

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1.	/ 8
2.	/ 8
3.	/ 8
4.	/ 6
5.	/ 12
TOTAL	/42

Values

[8] 1. Determine whether the series is absolutely convergent, conditionally convergent or divergent.

[2] (a) $\sum_{n=1}^{\infty}(-1)^n \frac{3^{1+3n}}{n^n}$

(b) $\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n^2+1}$

(c) $\sum_{n=1}^{\infty} \frac{(1.1)^n}{n^2}$

Solution. (a) Root test: $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{3(3^3)^n}{n^n}} = \lim_{n\to\infty}\sqrt[n]{3}\,\frac{3^3}{n} = (1)(0) = 0 < 1$ so the series converges absolutely.

(b) Ratio test fails; on the other hand $\lim_{n\to\infty}\frac{\frac{n+1}{n^2+1}}{\frac{1}{n}} = 1$, so that the series diverges absolutely. The series itelsef is alternating. The term goes to 0 and is decreasing (since $\left(\frac{x+1}{x^2+1}\right)' = \frac{x^2+1-2x^2-2x}{x^2+1} = \frac{-x^2+1-2x}{x^2+1}$ is obviously less than 0 for $x \geq 1$), so the alternating series test tells us that the series converges conditionally.

(c) Ratio test: $\lim_{n\to\infty}\frac{(1.1)^{n+1}n^2}{(n+1)^2(1.1)^n} = (1.1)\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1.1 > 1$, so the series diverges as is and absolutely.

Values

[8] 2. Find a power series representation of the function $f(x)=\dfrac{1}{1+9x^2}$ and find the interval of convergence.

Solution. Denote $g(x)=\dfrac{1}{1-x}$; then $g(x)=\sum_{n=0}^{\infty}x^n$ for x in $(-1,1)$. We see that $g(-9x^2)=f(x)$. So, $f(x)=\sum_{n=0}^{\infty}(-9x^2)^n=\sum_{n=0}^{\infty}(-1)^n 9^n x^{2n}$. This converges for $9x^2$ in $(-1,1)$; this means $-1<9x^2<1$, which makes *x* between $-\dfrac{1}{3}$ and $\dfrac{1}{3}$. So, the interval of convergence is $(-\dfrac{1}{3},\dfrac{1}{3})$.

[8] 3. (a) Use the ratio test to find the interval of convergence of the series $\sum_{n=1}^{\infty}nx^{n-1}$

(b) Compute the sum of the series $\sum_{n=1}^{\infty}\dfrac{n}{2^n}$. [Hint: fist find a closed form expression for the power series in part (a))

Solution. (a) $\lim_{n\to\infty}\dfrac{(n+1)|x|^n}{n|x|^{n-1}}=|x|\lim_{n\to\infty}\dfrac{(n+1)}{n}=|x|$, and so the series coverges over $(-1,1)$. It is easy to see by the divergence test that the series does not converge at the edges of this interval, so that $(-1,1)$ is the interval of convergence.

(b) Observe that $\sum_{n=1}^{\infty}nx^{n-1}=\left(\sum_{n=0}^{\infty}x^n\right)'$. On the other hand we know that $\sum_{n=0}^{\infty}x^n=\dfrac{1}{1-x}$ over $(-1,1)$. So $\sum_{n=1}^{\infty}nx^{n-1}=\left(\dfrac{1}{1-x}\right)'=\dfrac{1}{(1-x)^2}$. Now $\sum_{n=1}^{\infty}\dfrac{n}{2^n}=\dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{n}{2^{n-1}}=\dfrac{1}{2}\dfrac{1}{(1-\frac{1}{2})^2}=2$.

Values

[8] 4. (a) Find the Maclaurin series represenatation for the function $f(x)=\cos(x^2)$.

(b) Find $f^{(2011)}(0)$ (that is, the 2011$^{\text{th}}$ derivative of $f(x)=\cos(x^2)$ at the point $x=0$) and find $f^{(400)}(0)$.

Solution. (a) We know that $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$. So,

$$\cos(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(x^2\right)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}.$$

(b) $\frac{f^{(2011)}(0)}{2011!}$ is the coefficient in front of x^{2011} in the Maclaurin series representation for $f(x)$. In this case, as we see, this coefficient is 0, so $f^{(2011)}(0)=0$. Similarly, $\frac{f^{(400)}(0)}{400!}$ is the coefficient in front of x^{400}, which is $\frac{(-1)^{100}}{200!}=\frac{1}{200!}$. So $f^{(400)}(0)=\frac{400!}{200!}$.

[7] 5. Prove that $\sin x$ is equal to its Maclaurin series representation by showing that the reminder $R_n(x)$ tends to 0 as n tends to infinity.

Done in the textbook.

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