Values

[8] 1. (a) [2] State precisely the definition of a convergent sequence $\{a_n\}$ (i.e., what does it means to say that $\lim_{n\to\infty} a_n = L$.

[6] (b) Show that $\lim_{n\to\infty}\left(\frac{1}{e^n}+1\right)=1$ providing an argument based only on the definition of convergent sequence (as in part (a) of this question). No points will be given if other properties of sequences are used.

Solution. Given $\varepsilon > 0$ we are looking for N such that if $n > N$, then $\left|\frac{1}{e^n}+1-1\right| < \varepsilon$. Focusing on the last inequality:

$\left|\frac{1}{e^n}+1-1\right| < \varepsilon \Leftrightarrow \left|\frac{1}{e^n}\right| < \varepsilon \Leftrightarrow \frac{1}{e^n} < \frac{\varepsilon}{1} \Leftrightarrow e^n > \frac{1}{\varepsilon} \Leftrightarrow n > \ln\frac{1}{\varepsilon}$. Choose any N such that $N > \ln\frac{1}{\varepsilon}$. For such N we have: if $n > N$, then $n > \ln\frac{1}{\varepsilon}$, and backing up along $\Leftrightarrow$ we get $\left|\frac{1}{e^n}+1-1\right| < \varepsilon$, which is what we want.

Values

[8] 2.

Consider the sequence $\{a_n\}$ defined as follows: $a_1 = 1$, $a_n = \sqrt{1+a_{n-1}}$ for $n = 2,3,4,\ldots$.

(The first few terms of the sequence are as follows: 1, $\sqrt{1+\sqrt{1}}$, $\sqrt{1+\sqrt{1+\sqrt{1}}}$, $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}$)

This is an increasing sequence (do NOT show that the sequence is indeed increasing).

(a) [4] Show by means of induction that the sequence $\{a_n\}$ is bounded by 2.

Solution. This is obviously true for a_1. Assume $a_n < 2$. Trying to show that $a_{n+1} < 2$. We have $a_{n+1} < 2 \Leftrightarrow \sqrt{1+a_n} < 2 \Leftrightarrow 1 + a_n < 4 \Leftrightarrow a_n < 3$. Since we have assumed that $a_n < 2$, it has to be that $a_n < 3$. Backing up along $\Leftrightarrow$ we conclude that $a_{n+1} < 2$, as we wanted to show.

(b) [4] Is the sequence $\{a_n\}$ convergent? Why? If it is convergent, find $\lim_{n\to\infty} a_n$.

Solution. It is; theorem, the sequence is bounded from above and increasing. Suppose $\lim_{n\to\infty} a_n = x$. Then we have $x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{1+a_{n-1}} = \sqrt{1+x}$. Solve $x = \sqrt{1+x}$, means solve $x^2 - x - 1 = 0$, and the only positive solution is $x = \dfrac{1+\sqrt{5}}{2}$.

Values

[8] 3. Does the following series converge? Justify your answer, and if it converges, find the sum of the series. You may use theorems from class.

(a) $\sum_{n=1}^{\infty} 2^{5-n}$

(b) $\sum_{n=1}^{\infty} \frac{1-n^2}{1+2n+5n^2}$

Solution. **(a)** [5] Geometric series: $\sum_{n=1}^{\infty} 2^{5-n} = 2^5 \sum_{n=1}^{\infty} 2^{-n} = 2^5 \left(\frac{1}{1-\frac{1}{2}} - 1 \right) = 2^5$

(b) [3] $\lim_{n\to\infty} \frac{1-n^2}{1+2n+5n^2} = -\frac{1}{5} \neq 0$, so the series diverges by the divergence test.

[6] 4. First check that the integral test is applicable, then use it to determine is the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

Solution. We consider the function $f(x) = \frac{1}{x(\ln x)^2}$, $x \geq 2$. It is positive (since both x and $\ln x$ are positive for $x \geq 2$), it is continuous (since both x and $\ln x$ are continuous, and since both are not 0 for $x \geq 2$), and it is decreasing (since as x becomes larger, so does the denominator of $\frac{1}{x(\ln x)^2}$, so that $\frac{1}{x(\ln x)^2}$ is becoming smaller). So, the integral test is applicable.

$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t \to \infty} \int_2^t \frac{dx}{x(\ln x)^2}$; use the substitution $u = \ln x$; the integral changes to $\lim_{t \to \infty} \int_{\ln 2}^t \frac{du}{u^2}$. Simple computation then gives $\frac{1}{\ln 2}$ as the value of that integral. SO, the improper integral converges; so the series converges too.

[14] 6 Use the comparison test, or the limit comparison test to determine if the following series converges. You may use theorems from class.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n^2}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{n^2-2}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$

Solution.

(a) [4] It is obvious that $\dfrac{1}{n+n^2} < \dfrac{1}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (theorem), it follows by the comparison test that so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n^2}$.

(b) [4] $\displaystyle\lim_{n\to\infty} \frac{\frac{n-1}{n^2-2}}{\frac{1}{n}} = 1$ (by an easy computation). It follows from the limit comparison test and from the fact that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, that $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{n^2-2}$ diverges too.

(c) [4] This is an alternating series, so we try the alternating series test.

(i) We compute $\displaystyle\lim_{n\to\infty} \frac{n}{2^n}$ by switching to functions and computing $\displaystyle\lim_{x\to\infty} \frac{x}{2^x}$. We use L'Hospital rule in the first step: $\displaystyle\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$.

(ii) We can show that $\dfrac{n+1}{2^{n+1}} \le \dfrac{n}{2^n}$ directly, since this inequality simplifies to $n+1 \le 2n$, or $1 \le n$, which is obvious. (Alternatively, one can show using derivatives that the function $\dfrac{x}{2^x}$ decreases.)

It follows from the Alternating Series Test that the series in this problem converges.