FOURIER SERIES

Many of the elementary functions of mathematics can be written in the form of a power series, called a Taylor series. Periodic functions and discontinuous functions, which do not have a Taylor series representation, may be able to be represented by a trigonometric series called a Fourier series.

<u>Periodic Functions:</u> A function is said to be <u>periodic</u> if $f(x + p) = f(x)$ for all x. The least value of $p>0$ is called the <u>period</u> of $f(x)$. The sine and cosine are familiar examples of periodic functions. They are both periodic with period 2π.

<u>Fourier Series:</u> A series of the form $\frac{a_0}{2} + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)$ is called a <u>Fourier series</u>. The numbers a_n and b_n are called the <u>Fourier coefficients</u> of $f(x)$. Since all the terms in a Fourier series are periodic functions, a function representable by a Fourier series must be periodic. ?

<u>Determining the Fourier Coefficients: (Euler's Formulas)</u>

Let us suppose that $f(x)$ is a periodic function having a period 2π and that $f(x)$ can be represented by a Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx) \quad \dots\dots\dots(1)$$

We determine the coefficients a_n and b_n as follows. We first determine a_0 by integrating both sides of equation (1) from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left[\frac{a_0}{2} + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)\right] dx. \quad \dots\dots(2)$$

Assuming that the series converges to $f(x)$ uniformly on the interval $[-\pi, \pi]$, we may integrate equation (2) term-by-term. The result is

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + a_n \sum_1^{\infty} \int_{-\pi}^{\pi} \cos nx\,dx + b_n \sum_1^{\infty} \int_{-\pi}^{\pi} \sin nx\,dx. \cdots(3)$$

Now $\int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = \pi a_0$ while $\int_{-\pi}^{\pi} \cos nx\,dx = \frac{\sin nx}{n}\Big|_{-\pi}^{\pi} = 0 - 0 = 0 \quad \forall\ n>0$

and $\int_{-\pi}^{\pi} \sin nx\,dx = -\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} = 0$ for all positive integers n.

Substituting these results into eq'n(3) gives $\int_{-\pi}^{\pi} f(x)\,dx = \pi a_0 + 0 + 0$

One can therefore conclude that $\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$ ....

To determine a_m where $m \geq 1$ we multiply both sides of equation (1) by cos mx where m is a fixed positive integer and integrate term-by-term from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx\,dx + a_n\sum_{1}^{\infty}\int_{-\pi}^{\pi}\cos nx\cos mx\,dx$$

$$+\, b_n\sum_{1}^{\infty}\int_{-\pi}^{\pi}\sin nx\cos mx\,dx. \quad \cdots\cdots\cdots(5)$$

Now $\int_{-\pi}^{\pi}\cos mx\,dx = \left.\frac{\sin mx}{m}\right|_{-\pi}^{\pi} = 0 - 0 = 0$ while

$$\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}\cos(n+m)x\,dx + \int_{-\pi}^{\pi}\frac{1}{2}\cos(n-m)x\,dx = \begin{cases}0 & n\neq m\\ \pi & n=m\end{cases}$$

and $\int_{-\pi}^{\pi}\sin nx\cos mx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}\sin(n+m)x\,dx + \int_{-\pi}^{\pi}\frac{1}{2}\sin(n-m)x\,dx = 0+0=0.$

Substituting these results into (5) gives $\int_{-\pi}^{\pi} f(x)\cos mx\,dx = 0 + \pi a_m + 0$

We can therefore conclude $\boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx}$ ....(6)

To determine b_m we multiply both sides of equation (1) by sin mx where m is a fixed positive integer and integrate term-by-term from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin mx\,dx + a_n\sum_{1}^{\infty}\int_{-\pi}^{\pi}\cos nx\sin mx\,dx$$

$$+\, b_n\sum_{1}^{\infty}\int_{-\pi}^{\pi}\sin nx\sin mx\,dx. \quad \cdots\cdots\cdots\cdots(7)$$

Now $\int_{-\pi}^{\pi}\sin mx\,dx = -\left.\frac{\cos mx}{m}\right|_{-\pi}^{\pi} = 0$ and $\int_{-\pi}^{\pi}\cos nx\sin mx\,dx = 0$ as before.

$$\int_{-\pi}^{\pi}\sin nx\sin mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)x\,dx - \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)x\,dx = \begin{cases}0 & n\neq m\\ \pi & n=m\end{cases}$$

Substituting into equation (7) gives $\int_{-\pi}^{\pi} f(x)\sin mx\,dx = 0 + 0 + \pi b_m$.

We can therefore conclude $\boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx}$ ....(8)

Replacing m by n in equations (6) and (8) and combining equations (4) and (6) we get the following formulas for a_n and b_n.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad n = 0, 1, 2, 3, \ldots\ldots\ldots\ldots\ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots$$

Odd and Even Functions: A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$. Examples of odd functions are x^3 and $\sin x$. A function is said to be an even function if $f(-x) = f(x)$. Examples of even functions are x^2 and $\cos x$. The Fourier series of an odd function contains only sine terms while the Fourier series of an even function contains only cosine terms and possibly a constant term.

A function $f(x)$ is said to be piecewise continuous in the interval $[a, b]$ if it has only a finite number of finite discontinuities in the interval $[a, b]$.

If $f(x)$ is a periodic function with period 2π and $f(x)$ is piecewise continuous on the interval $[-\pi, \pi]$ and if $f(x)$ has a left and right hand derivative at each point of the interval, then the Fourier series for $f(x)$ is convergent. If $f(x)$ is continuous at x_0, then Fourier series converges to $f(x_0)$ at $x = x_0$. If $f(x)$ has a discontinuity at $x = x_0$, the Fourier series converges to the average of the left and right-hand limits of $f(x)$ at x_0.

Example: Find the Fourier series for the function

$$f(x) = |x|, \quad -\pi \le x \le \pi, \qquad f(x + 2\pi) = f(x).$$

Solution: Since $f(x)$ is an even function, $b_n = 0$ for all n.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right] = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[\frac{\cos nx}{n^2} + \frac{x\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \begin{cases} 0 & \text{if n is even} \\ -\dfrac{4}{\pi n^2} & \text{if n is odd} \end{cases}$$

The Fourier series is $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

Setting $x = 0$ above and noting that $f(0) = 0$ gives the interesting formula $\pi^2 = 8\left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\cdots\cdots\right]$

PROBLEMS

1. Find the Fourier series for $f(x) = x, -\pi<x<\pi, f(x+2\pi)=f(x)$.

2. (a) Find the Fourier series for $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$

   (b) Set $x = \frac{\pi}{2}$ in your result from part (a) to derive an expression for π.

3. (a) Find the Fourier series for $f(x) = x^2, -\pi<x<\pi, f(x+2\pi)=f(x)$.

   (b) Set $x = \pi$ in the Fourier series for x^2 to get a formula for π^2.

   (c) Use the result of 3(b) to find a formula for $\sum_1^\infty \frac{1}{n^2}$

4. Find the Fourier series for $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi & 0 < x < \pi \end{cases}$

5. Find the Fourier series for $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

ANSWERS

1. $f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\cdots\cdots\cdots\right]$

2. (a) $f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\cdots\cdots\cdots\right]$

   (b) $\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots\cdots\cdots\right]$

3. (a) $f(x) = \frac{\pi^2}{3} - 4\left[\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \frac{\cos 4x}{16} + \cdots\cdots\cdots\right]$

   (b) $\pi^2 = 6\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\cdots\cdots\cdots\right]$

   (c) $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$

4. $f(x) = \frac{\pi}{2} + 2\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\cdots\cdots\cdots\right]$

5. $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\cdots\cdots\cdots\right]$

   $+ \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\cdots\right]$