MATH 2730 Sequences and Series
Midterm Exam
5:30-6:30, February 27, 2008

Brief Solutions

1. **(a)** Complete the following definition: A sequence $\{a_n\}_{n=1}^{\infty}$ converges to a number L (written $\lim_{n\to\infty} a_n = L$) if ….

(b) Use only the definition of the limit of a sequence to show that the sequence $\left\{\frac{-1}{\sqrt{n}+1}\right\}_{n=1}^{\infty}$ converges to 0 (i.e., that $\lim_{n\to\infty}\frac{-1}{\sqrt{n}+1}=0$). No points will be given if the definition of *limit* is not used.

Given $\varepsilon > 0$, we are looking for N such that if $n > N$ then $\left|\frac{-1}{\sqrt{n}+1}-0\right| < \varepsilon$. We analyze the last inequality:

$$\left|\frac{-1}{\sqrt{n}+1}-0\right| < \varepsilon \Leftrightarrow \left|\frac{-1}{\sqrt{n}+1}\right| < \varepsilon \Leftrightarrow \frac{1}{\sqrt{n}+1} < \varepsilon \Leftrightarrow 1 < \varepsilon(\sqrt{n}+1) \Leftrightarrow 1 < \varepsilon\sqrt{n}+\varepsilon \Leftrightarrow$$

$$\Leftrightarrow 1-\varepsilon < \varepsilon\sqrt{n} \Leftrightarrow \frac{1-\varepsilon}{\varepsilon} < \sqrt{n} \Leftrightarrow \left(\frac{1-\varepsilon}{\varepsilon}\right)^2 < n \qquad (*)$$

(We have assumed in the last step that $\varepsilon < 1$, but that is a minor point. Each of the steps indicated by $\Leftrightarrow$ is simple or evident.)

Now we **choose** N to be any number larger than $\left(\frac{1-\varepsilon}{\varepsilon}\right)^2$. Then **if** $n > N$, and the last inequality in the above long sequence of inequalities indicated by (*) will be satisfied. Consequently, so will the first inequality in (*), and so we achieved what we wanted.

2. Check if the following series converges, and if it does converge, find its sum.

(a) $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n}\sqrt{n+1}}$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n}\sqrt{n^2+1}} = \lim_{n\to\infty}\frac{n}{\sqrt{n}\sqrt{n}\sqrt{1+\frac{1}{\sqrt{n}}}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}} = 1 > 0$$ and so the series diverges by the divergence test.

(b) $\sum_{n=1}^{\infty}\frac{2}{n(n+2)}$

$\frac{2}{n(n+2)} = \frac{A}{n}+\frac{B}{n+2}$ gives $2 = A(n+2)+Bn$ which in turn yields the system $2 = 2A$, $0 = A+B$. Solving, yields $A=1$, $B=-1$, so that $\frac{2}{n(n+2)} = \frac{1}{n}-\frac{1}{(n+2)}$. Now we write the n-th partial sum of the series:

$$s_n = (1-\frac{1}{3}) + (\frac{1}{2}-\frac{1}{4}) + (\frac{1}{3}-\frac{1}{5}) + (\frac{1}{4}-\frac{1}{6}) + ...$$

$$... + (\frac{1}{n-3}-\frac{1}{n-1}) + (\frac{1}{n-2}-\frac{1}{n}) + (\frac{1}{n-1}-\frac{1}{n+1}) + (\frac{1}{n}-\frac{1}{n+2})$$

We see that almost all of the terms cancel, and we get that $s_n = 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$. Now it is easy to calculate $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}) = \frac{3}{2}$. So the series converges and its sum is $\frac{3}{2}$.

(c) $\sum_{n=0}^{\infty}\frac{2^n}{3^{n+2}}$

$$\sum_{n=0}^{\infty}\frac{2^n}{3^{n+2}} = \sum_{n=0}^{\infty}\frac{2^n}{9(3^n)} = \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{9}\frac{1}{1-\frac{2}{3}}.$$

3. **(a)** First confirm that the integral test can be used for the series $\sum_{n=2}^{\infty}\frac{1234}{n(\ln n)^{2008}}$, then use it to check if that series converges.

First of all, $\sum_{n=2}^{\infty}\frac{1234}{n(\ln n)^{2008}} = 1234\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{2008}}$, so that we can focus on the last series.
Consider the function $f(x) = \frac{1}{x(\ln x)^{2008}}$ for $x \geq 2$. It is obviously positive, continuous and decreasing. So, the integral test is usable.

$\int_2^{\infty}\frac{dx}{x(\ln x)^{2008}} = \lim_{t\to\infty}\int_2^{t}\frac{dx}{x(\ln x)^{2008}}$. The substitution $u = \ln x$ gives

$$\lim_{t\to\infty}\int_2^{t}\frac{dx}{x(\ln x)^{2008}} = \lim_{t\to\infty}\int_{x=2}^{x=t}\frac{du}{u^{2008}} = \lim_{t\to\infty}(\frac{u^{-2007}}{-2007}\Big|_{x=2}^{x=t}) = \lim_{t\to\infty}(\frac{(\ln x)^{-2007}}{-2007}\Big|_2^t) =$$

$$= \lim_{t\to\infty}(\frac{1}{2007(\ln 2)^{2007}} - \frac{1}{2007(\ln t)^{2007}}) = \frac{1}{2007(\ln 2)^{2007}}$$

Since we got a finite number, the integral converges, and so does the series.

(b) Check if the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+2^n}$ converges.

This is an alternating series. Putting $b_n = \frac{1}{n^2+2^n}$, we see that this sequence is decreasing and that $\lim_{n\to\infty} b_n = 0$. So, the series converges by the alternating series test.

4. Check if the series converges conditionally, if it converges absolutely or if it diverges.

(a) $\sum_{n=1}^{\infty}\frac{2\cos(\pi n)}{2n+3}$

Since $\cos(\pi n)$ is 1 for even n and –1 for odd n, this is an alternating series. The alternating series test applies and it is easy to see that it shows that the series converges. However, the series $\sum_{n=1}^{\infty} \frac{2}{2n+3}$ made of the absolute values of the terms diverges: that is also easy to see by, say, using the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$. We conclude that the series given in the statement of this question converges conditionally.

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

This is a positive series, so, for this series, the notion of absolute convergence is the same as the notion of convergence. We use the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1}{(n+1)(n+1)} = 4$$

and since this number is >1, the series diverges.