# Math 2730 Assignment 4

# Solutions

1. Find the power series with sum equal to $g(x) = \dfrac{x^2}{(1+2x)^2}$ and find the interval of convergence of the power series.

*Solution*. First we focus on the function $h(x) = \dfrac{1}{(1+2x)^2}$. Observe first that $h(x) = \dfrac{1}{2}\left(\dfrac{1}{1+2x}\right)'$. Now, from what we know, $\dfrac{1}{1+2x} = \sum_{n=0}^{\infty}(-2x)^n$ and the series converges for $|2x| < 1$, which gives $-\dfrac{1}{2} < x < \dfrac{1}{2}$. Consequently, $h(x) = \dfrac{1}{2}\left(\dfrac{1}{1+2x}\right)' = \dfrac{1}{2}\left(\sum_{n=0}^{\infty}(-2x)^n\right)'$ and within $-\dfrac{1}{2} < x < \dfrac{1}{2}$ we can differentiate term by term to get $\dfrac{1}{2}\left(\sum_{n=0}^{\infty}(-2x)^n\right)' = \dfrac{1}{2}\sum_{n=1}^{\infty}(-2)^n nx^{n-1}$ (note the change of the start of the counter: the term we get for n=0 in the series we differentiate is annihilated after differentiation). So, summarizing, $h(x) = \dfrac{1}{2}\sum_{n=1}^{\infty}(-2)^n nx^{n-1}$ for $-\dfrac{1}{2} < x < \dfrac{1}{2}$. Our original function was $x^2h(x)$, and so $x^2h(x) = \dfrac{1}{2}x^2\sum_{n=1}^{\infty}(-2)^n nx^{n-1} = \sum_{n=1}^{\infty}(-2)^{n-1}nx^{n+1}$ over $-\dfrac{1}{2} < x < \dfrac{1}{2}$ (where in the second equality we have simply multiplied every term of the series by $\dfrac{1}{2}x^2$.

**2.** Show directly (using Taylor's inequality) that $\cos x = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{(2n)!}$.

*Solution*. Every derivative of $\cos x$ is either $\pm\cos x$ or $\pm\sin x$, and so, with $f(x) = \cos x$, we have that $\left|f^{(n+1)}(x)\right| \le 1$ for every *x*. Consequently, by Taylor's inequality, $|R_n(x)| \le \dfrac{|x|^{n+1}}{(n+1)!}$ for every *x*. Since $\lim_{n\to\infty}\dfrac{|x|^{n+1}}{(n+1)!} = 0$, it follows that $\lim_{n\to\infty} R_n = 0$ and so $f(x) = \cos x$ is equal to its Maclaurin series representation.

**3.** Find the Maclaurin series representation for the following functions and identify the interval of convergence of the series.

**(a)** $\dfrac{e^{2x^2}-1}{x^2}$

**(b)** $\sin x\cos x$ (Hint: start with $\sin 2x$)

**(c)** $\tan^{-1}(3x)$

**(a)** We know that $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all numbers *x*. So $e^{2x^2}=\sum_{n=0}^{\infty}\frac{(2x^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{2^n x^{2n}}{n!}$ and thus $\frac{e^{2x^2}-1}{x^2}=\frac{1}{x^2}\left(\sum_{n=0}^{\infty}\frac{2^n x^{2n}}{n!}-1\right)=\frac{1}{x^2}\left(\sum_{n=1}^{\infty}\frac{2^n x^{2n}}{n!}\right)$ where in the last equality we have cancelled the term we get for n=0 with the –1). Further, $\frac{1}{x^2}\sum_{n=1}^{\infty}\frac{2^n x^{2n}}{n!}=\sum_{n=1}^{\infty}\frac{2^n x^{2n-2}}{n!}$ (after multiplying each term by $\frac{1}{x^2}$). So $\frac{e^{2x^2}-1}{x^2}=\sum_{n=1}^{\infty}\frac{2^n x^{2n-2}}{n!}$ and the representation is true for every number *x* except *x*=0.

**(b)** Recall that $\sin 2x=2\sin x\cos x$ so that our starting function $\sin x\cos x$ is the same as $\frac{1}{2}\sin 2x$. We know that $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for every number *x*, so that $\frac{1}{2}\sin 2x=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}x^{2n+1}}{(2n+1)!}$.The representation is true for every number x.

**(c)** We have that $(\tan^{-1}x)'=\frac{1}{1+x^2}$. On the other hand, since $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ for *x* in (-1,1), it follows that $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}$ over the same interval (-1,1). Consequently, $\tan^{-1}x=\int\frac{1}{1+x^2}dx+c=\int\sum_{n=0}^{\infty}(-1)^n x^{2n}+c$. Within the interval of convergence we can integrate term by term, so that $\tan^{-1}x=\int\left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right)dx+c=\sum_{n=0}^{\infty}\int(-1)^n x^{2n}dx+c=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}+c$. Since $\tan^{-1}0=0$ and since the series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ is also 0 when *x*=0, it follows that the constant *c* is 0. So, $\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ for *x* in the interval (-1,1), and so

$\tan^{-1} 3x = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{2n+1}$ for $3x$ in (-1,1), i.e., for $x$ in the interval $(-\frac{1}{3}, \frac{1}{3})$.

**4.** Find the Taylor series representation of the function $\ln x$ centered at a=3.

Taylor's formula tells us that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. In this question $f(x) = \ln x$ and *a*=3. We compute: $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2\cdot 1}{x^3}$, $f^{(4)}(x) = -\frac{3\cdot 2\cdot 1}{x^4}$, and, in general, $f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$. So, $f^{(n)}(3) = (-1)^{n+1}\frac{(n-1)!}{3^n}$ (with $f^{(0)}(3) = f(3) = \ln 3$) and, after substituting this in the Taylor's formula, we get

$f(x) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\frac{(n-1)!}{3^n}}{n!}(x-3)^n$. We simplify this a bit to get our final answer:

$f(x) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n}(x-3)^n$.

**4.** Use multiplication of series to find the first three nonzero terms of the Maclaurin series representation of the function $\ln(2+x)\cdot \tan^{-1}\left(x^2\right)$.

First, $\ln(1+x) = \int \frac{1}{1+x}dx + c = \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx + c$, where we have used the fact that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for –1<*x*<1. Continuing,

$\int\left(\sum_{n=1}^{\infty}(-1)^n x^n\right)dx + c = \sum_{n=0}^{\infty}\int (-1)^n x^n dx \; + c = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \; + c$. So that

$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + c$. Substituting *x*=0, we get that c=0. So

$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$. Now, $\ln(2+x) = \ln[2(1+x/2)] = \ln 2 + \ln(1+x/2)$ and since

$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$, we have that

$\ln(1+x/2) = \sum_{n=0}^{\infty}(-1)^n \frac{(x/2)^{n+1}}{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{2^{n+1}(n+1)}$. Summarizing,

$\ln(2+x) = \ln 2 + \ln(1+x/2) = \ln 2 + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{2^{n+1}(n+1)}$.

Now we pay attention to $\tan^{-1}\left(x^2\right)$. Since $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$, we get that

$$\tan^{-1}\left(x^2\right) = \sum_{n=0}^{\infty}(-1)^n \frac{\left(x^2\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}.$$

Finally we take a look at the product: $\ln(2+x)\cdot\tan^{-1}\left(x^2\right)$:

$$\ln(2+x)\cdot\tan^{-1}\left(x^2\right) = \left[\ln 2 + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{2^{n+1}(n+1)}\right]\cdot\left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}\right] =$$

$$= \left[\ln 2 + \frac{x}{2} - \frac{x^2}{2^3} + \ldots\right]\left[x^2 - \frac{x^6}{3} + \ldots\right] = (\ln 2)x^2 - \frac{x^3}{2} - \frac{x^4}{2^3} + \ldots$$

**5.** Use power series to evaluate $\int_0^x \cos(t^2)\,dt$.

Since $\cos t = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}$, we have that $\cos t^2 = \sum_{n=0}^{\infty}\frac{(-1)^n \left(t^2\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n t^{4n}}{(2n)!}$. So,

$$\int_0^x \cos(t^2)\,dt = \int_0^x \sum_{n=0}^{\infty}\frac{(-1)^n t^{4n}}{(2n)!}dt = \sum_{n=0}^{\infty}\int_0^x \frac{(-1)^n t^{4n}}{(2n)!}\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}.$$