MATH 2730 Assignment 3

Due March 10, 2008, (Solutions)

1. Which of the following series converges absolutely, which converges conditionally and which diverges? Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(0.1)^n}{n}$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-2)^{n+1}}{n+5^n}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\sqrt[n]{10}$

(d) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\left(\frac{\ln n}{\ln(n^2)}\right)^n$

(a) $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}(0.1)^n}{n}\right|=\sum_{n=1}^{\infty}\frac{(0.1)^n}{n}=\sum_{n=1}^{\infty}\frac{1}{n(10)^n}$ and since $\displaystyle\frac{1}{n(10)^n}<\frac{1}{(10)^n}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(10)^n}$ converges, we have by the comparison test that $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}(0.1)^n}{n}\right|$ converges. Consequently the series in this problem converges absolutely.

(b) $\displaystyle\sum_{n=2}^{\infty}\left|\frac{(-2)^{n+1}}{n+5^n}\right|=\sum_{n=2}^{\infty}\frac{2^{n+1}}{n+5^n}=2\sum_{n=2}^{\infty}\frac{2^n}{n+5^n}$, since $\displaystyle\frac{2^n}{n+5^n}<\frac{2^n}{5^n}=\left(\frac{2}{5}\right)^n$ and since $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2}{5}\right)^n$ converges, the original series converges absolutely.

(c) $\displaystyle\lim_{n\to\infty}\sqrt[n]{10}=\lim_{n\to\infty}10^{\frac{1}{n}}=10^0=1$ and so $\displaystyle\lim_{n\to\infty}(-1)^{n+1}\sqrt[n]{10}$ is not 0. Thereby the series diverges (by the divergence test).

(d) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\left(\frac{\ln n}{\ln(n^2)}\right)^n=\sum_{n=2}^{\infty}(-1)^n\left(\frac{\ln n}{2\ln n}\right)^n=\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n$ and since $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^n$ converges, the series in this problem converges absolutely.

2. Find the interval of convergence for the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$

(b) $\sum_{n=0}^{\infty}\frac{(2x+3)^{2n+1}}{n!}$

(c) $\sum_{n=1}^{\infty}(n)^n x^n$

For the series $\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$ (in part **a** above), fund the sum of the series as a function on x.

(a) We use the ratio test (the root test is also easy to use here):

$\lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{10^{n+1}}}{\frac{(x-2)^n}{10^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)}{10}\right| = \left|\frac{(x-2)}{10}\right|$ and so the series converges when $\left|\frac{(x-2)}{10}\right| < 1$, which, after solving yields $-8 < x < 12$. When $x = 12$ the series becomes $\sum_{n=0}^{\infty}\frac{(12-2)^n}{10^n} = \sum_{n=0}^{\infty}1$ which obviously diverges, while when $x = -8$ the series becomes $\sum_{n=0}^{\infty}\frac{(-8-2)^n}{10^n} = \sum_{n=0}^{\infty}(-1)^n$ which also diverges. Consequently, the interval of convergence of this series is $(-8\ ,\ 12)$.

We immediately take care of the last part of this problem and find the sum of this series over the interval of convergence: $\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n} = \sum_{n=0}^{\infty}\left(\frac{x-2}{10}\right)^n = \frac{1}{1-\frac{x-2}{10}}$.

(b) Use the ratio test again: $\lim_{n\to\infty}\left|\frac{\frac{(2x+3)^{2n+3}}{(n+1)!}}{\frac{(2x+3)^{2n+1}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(2x+3)^2}{n+1}\right| = 0$ and since that is always less than 1 the series $\sum_{n=0}^{\infty}\frac{(2x+3)^{2n+1}}{n!}$ converges always and the interval of convergence if $(-\infty, +\infty)$.

(c) Use the root test: $\lim_{n\to\infty}\sqrt[n]{|(nx)^n|} = \lim_{n\to\infty}|nx| = \infty$ with the last equality being true for all x except for x=0. Since the limit is never less than 1 for x not equal to 0, the series $\sum_{n=1}^{\infty}(n)^n x^n$ converges only when x=0 and the interval of convergence is [0,0].