136.271 Assignment 2

Solutions

Note before you start: there are many ways to solve the problems below, and I do not claim the solutions below are the shortest. They are just the first to come.

1. Use the integral test, the (simple) comparison test, the limit comparison test or the rest of the theory we have covered so far (first 4 sections) to check if the following series converges or diverges. (If you want to use the integral test, then you first need to show it is applicable.)

(a) $5+\frac{2}{3}+1+\frac{1}{7}+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}...$

Ignore the first few term and look at the series $\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}...$, i.e., the series $\sum_{n=2}^{\infty}\frac{1}{n!}$. Since $\frac{1}{n!}\leq\frac{1}{n(n-1)}$ and since $\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$ converges (an exercise done in class), it follows that $\sum_{n=2}^{\infty}\frac{1}{n!}$ converges. So, the original series (being different from $\sum_{n=2}^{\infty}\frac{1}{n!}$ by a few finite numbers) also converges.

(b) $\sum_{n=2}^{\infty}\frac{\ln n}{\sqrt{2n}}$

Obviously $\frac{\ln n}{\sqrt{2n}}>\frac{1}{\sqrt{2n}}$ for n large enough (larger than 2). The series $\sum_{n=3}^{\infty}\frac{1}{\sqrt{2n}}=\frac{1}{\sqrt{2}}\sum_{n=3}^{\infty}\frac{1}{\sqrt{n}}$ diverges (theorem). So, by the comparison test, the series $\sum_{n=3}^{\infty}\frac{\ln n}{\sqrt{2n}}$ also diverges. Consequently, the same is true for the series $\sum_{n=2}^{\infty}\frac{\ln n}{\sqrt{2n}}$.

(c) $\sum_{n=1}^{\infty}\frac{1+\cos n}{n^2}$

Since $-1\leq\cos n\leq 1$ it follows that $0\leq\frac{1+\cos n}{n^2}\leq\frac{2}{n^2}$. Since $\sum_{n=1}^{\infty}\frac{2}{n^2}=2\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (theorem), it follows by the comparison test that $\sum_{n=1}^{\infty}\frac{1+\cos n}{n^2}$ converges too.

(d) $\sum\limits_{n=1}^{\infty}\frac{(\ln n)^2}{n^{2/3}}$

We use the comparison test and compare with the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^{2/3}}$. Since $\frac{(\ln n)^2}{n^{2/3}} > \frac{1}{n^{2/3}}$ for n larger than 2, and since $\sum\limits_{n=3}^{\infty}\frac{1}{n^{2/3}}$ diverges (theorem), it follows that $\sum\limits_{n=3}^{\infty}\frac{(\ln n)^2}{n^{2/3}}$ diverges too. Consequently, so does $\sum\limits_{n=1}^{\infty}\frac{(\ln n)^2}{n^{2/3}}$.

(e) $\sum\limits_{n=3}^{\infty}\frac{\left(1/n\right)}{(\ln n)\sqrt{(\ln^2 n)-1}}$

Solution 1. We use the integral test. Consider the function $f(x)=\frac{1}{x(\ln x)\sqrt{(\ln^2 x)-1}}$ for x≥3. It is positive, decreasing and continuous and $f(n)=\frac{1}{n(\ln n)\sqrt{(\ln^2 n)-1}}$ over the integers larger than 2 (all of these claims are obvious in this case). So, the series converges if and only if the improper integral $\int\limits_3^{\infty}\frac{1}{x(\ln x)\sqrt{(\ln^2 x)-1}}dx$ converges. We use the substitution $\ln x = u$: $\int\limits_3^{\infty}\frac{1}{x(\ln x)\sqrt{(\ln^2 x)-1}}dx=\lim\limits_{a\to\infty}\int\limits_{x=3}^{x=a}\frac{1}{u\sqrt{u^2-1}}du$ and then $v=u^2-1$ $\lim\limits_{a\to\infty}\int\limits_{x=3}^{x=a}\frac{1}{u\sqrt{u^2-1}}du=\lim\limits_{a\to\infty}\int\limits_{x=3}^{x=a}\frac{u}{u^2\sqrt{u^2-1}}du=\frac{1}{2}\lim\limits_{a\to\infty}\int\limits_{x=3}^{x=a}\frac{dv}{(v+1)\sqrt{v}}$. Now keep in mind that our goal is to see if this integral converges. Take a look at the function $\frac{1}{(v+1)\sqrt{v}}$ and compare it with $\frac{1}{v\sqrt{v}}$: clearly the former is smaller (since we divide by more). Now, if we prove that $\frac{1}{2}\lim\limits_{a\to\infty}\int\limits_{x=3}^{x=a}\frac{dv}{v\sqrt{v}}$ is less than infinity, so will be the integral of the smaller function, i.e., so will the integral $\frac{1}{2}\lim\limits_{a\to\infty}\int\limits_{x=3}^{x=a}\frac{dv}{(v+1)\sqrt{v}}$. So, we focus on $\frac{1}{2}\lim\limits_{a\to\infty}\int\limits_{x=3}^{x=a}\frac{dv}{v\sqrt{v}}$: $\frac{1}{2}\lim\limits_{a\to\infty}\int\limits_{x=3}^{x=a}\frac{dv}{v\sqrt{v}}=\frac{1}{2}\lim\limits_{a\to\infty}(\frac{-2}{\sqrt{v}})\Bigg|_{x=3}^{x=a}=\frac{1}{2}\lim\limits_{a\to\infty}(\frac{-2}{\sqrt{u^2-1}})\Bigg|_{x=3}^{x=a}=\frac{1}{2}\lim\limits_{a\to\infty}(\frac{-2}{\sqrt{(\ln x)^2-1}})\Bigg|_{3}^{a}$, and, a bit more: $\frac{1}{2}\lim\limits_{a\to\infty}(\frac{-2}{\sqrt{(\ln x)^2-1}})\Bigg|_{3}^{a}=\lim\limits_{a\to\infty}(\frac{-1}{\sqrt{(\ln a)^2-1}}-\frac{-1}{\sqrt{(\ln 3)^2-1}})=\frac{1}{\sqrt{(\ln 3)^2-1}}$, and so, it does converge. We conclude (as we have explained above) that $\frac{1}{2}\lim\limits_{a\to\infty}\int\limits_{x=3}^{x=a}\frac{dv}{(v+1)\sqrt{v}}$ converges, so that the series converges too.

Solution 2. Use the limit comparison test with $\sum_{n=3}^{\infty} \frac{\left(\frac{1}{n}\right)}{(\ln n)\sqrt{(\ln^2 n)}} = \sum_{n=3}^{\infty} \frac{\left(\frac{1}{n}\right)}{\ln^2 n}$:

$\lim_{n\to\infty} \frac{\frac{\left(\frac{1}{n}\right)}{(\ln n)\sqrt{(\ln^2 n)-1}}}{\frac{\left(\frac{1}{n}\right)}{\ln^2 n}} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{(\ln^2 n)-1}} = 1$, and so the series $\sum_{n=3}^{\infty} \frac{\left(\frac{1}{n}\right)}{(\ln n)\sqrt{(\ln^2 n)-1}}$ and $\sum_{n=3}^{\infty} \frac{\left(\frac{1}{n}\right)}{\ln^2 n}$ converge or diverge together. Now use the integral test on the series $\sum_{n=3}^{\infty} \frac{\left(\frac{1}{n}\right)}{\ln^2 n}$ to conclude (after some short work) that it converges. So, the series $\sum_{n=3}^{\infty} \frac{\left(\frac{1}{n}\right)}{(\ln n)\sqrt{(\ln^2 n)-1}}$ also converges.

(f) $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$

Used the comparison test with $\sum_{n=2}^{\infty} \frac{1}{n}$: we show that $\frac{1}{1+\ln n} > \frac{1}{n}$ and the divergence $\sum_{n=2}^{\infty} \frac{1}{1+\ln n}$ (and so, of the series in this problem) would follow from the fact that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges. Note that $\frac{1}{1+\ln n} > \frac{1}{n}$ is equivalent to $n > 1+\ln n$, for n≥2, i.e., that $n-1-\ln n > 0$ for n≥2. Consider the function $f(x) = x-1-\ln x$ for x≥2. It is easy to see that $f(2) = 2-1-\ln 2$ is larger than 0. Now we show that the function increases, and the inequality will follow at once: $f'(x) = 1-\frac{1}{x} = \frac{x-1}{x}$ which is obviously larger than 0 since x is larger than 2. We have proven that $x-1-\ln x > 0$ for x≥2, and so $n-1-\ln n > 0$ for n≥2, as claimed.

Alternative solution: Apply the limit comparison test to the given series and the series $\sum_{n=3}^{\infty} \frac{\left(\frac{1}{n}\right)}{\ln^2 n}$.

2. Show that if $\sum_{n=1}^{\infty} a_n$ is a positive convergent series, then so is the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$.

Since $\frac{a_n}{n} \le a_n$ for all *n* the claim in this exercise follows from the simple comparison test.