136.272 Solutions

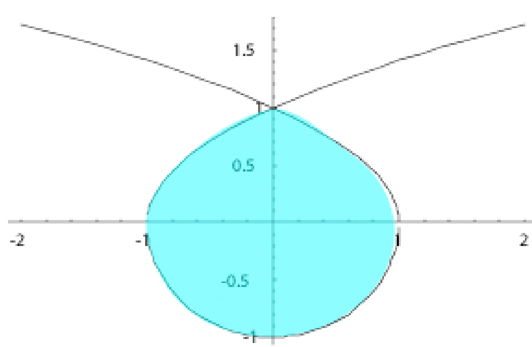
Assignment 4 (Sections 16.1-16.4)

1. [6 marks] Use the method of Lagrange multipliers to find and classify the extrema of the function $f(x,y)=xy$ subject to the constraint $x^2+y^2-4=0$.

Solution. Denote $F(x,y,\lambda)=f(x,y)+\lambda(x^2+y^2-4)=xy+\lambda(x^2+y^2-4)$. The equations $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$ and $\frac{\partial F}{\partial \lambda}=0$ are $y+2\lambda x=0$, $x+2\lambda y=0$ and $x^2+y^2-4=0$. Solving this gives 4 solutions for x and y: $(-\sqrt{2},-\sqrt{2})$, $(\sqrt{2},-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$ and $(\sqrt{2},\sqrt{2})$. We evaluate $f(x,y)=xy$ at these 4 points to get $f(-\sqrt{2},-\sqrt{2})=2$, $f(\sqrt{2},-\sqrt{2})=-2$, $f(-\sqrt{2},\sqrt{2})=-2$ and $f(\sqrt{2},\sqrt{2})=2$. So, *f* attains its absolute minimum of -2 at $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$, while it attains its absolute maximum of 2 at $(-\sqrt{2},-\sqrt{2})$ and $(\sqrt{2},\sqrt{2})$. [This is a consequence of the theorem regarding absolute extrema of functions over closed bounded domains, which is the case with the set of all points on the circle $x^2+y^2-4=0$.]

2. [6 marks] Evaluate $\iint_D (4xy^3-4x^2y)\,dA$ where D is the region bounded by $y=-\sqrt{1-x^2}$, $y=\sqrt{1-x}$ and $y=\sqrt{1+x}$. Sketch D.

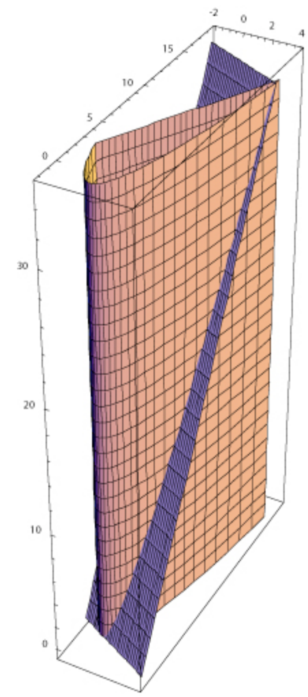
Solution. The parabolas $y=\sqrt{1-x}$ and $y=\sqrt{1+x}$ intersect at the point (0,1) (we get by solving the system of the two equations). The region D is shown in the picture below. We can describe it as follows: $-1\le x\le 0$ and $-\sqrt{1-x^2}\le y\le\sqrt{1+x}$ for the part of D to the left of the y-axis, and $0\le x\le 1$ and $-\sqrt{1-x^2}\le y\le\sqrt{1-x}$ for the part of D to the right of the y-axis.



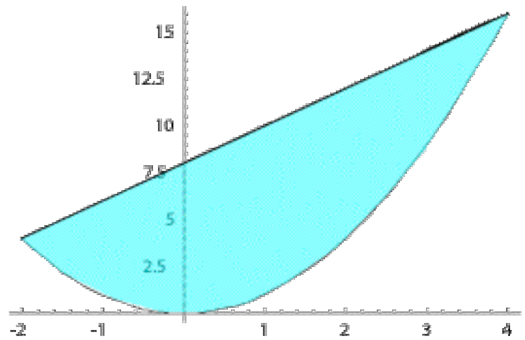
So we have $\iint_D (4xy^3-4x^2y)\,dA=\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1+x}}(4xy^3-4x^2y)\,dy\,dx+\int_{0}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x}}(4xy^3-4x^2y)\,dy\,dx=$

= (simple computations in between) = 1/5

3. [7 marks] Find the volume V of the solid S bounded by the xy-plane, the cylinder $y = x^2$, and the planes $z = x + 2y$ and $y = 2x + 8$. Sketch S.



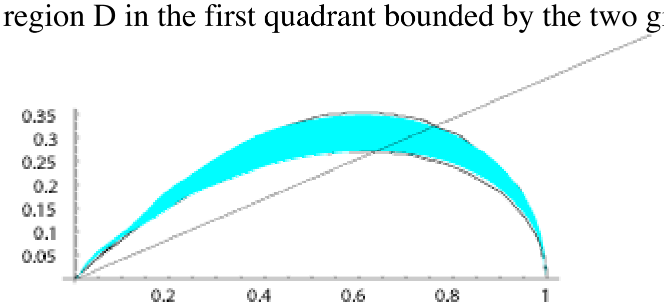
The solid S is show in the (3D) picture to the right. It is bounded above by the plane $z = x + 2y$ and below by the region B bounded by the curves $y = x^2$ and $y = 2x + 8$ in the xy-plane (shown in the picture below).



In the intersection of the two curves bounded the above region we have (after solving the system $y = x^2$, $y = 2x + 8$) that x=-2 (for the left hand side point) and x=4 (for the right hand side point). So, the volume we want is $\iint_B (x + 2y)dA = \int_{-2}^{4} \int_{x^2}^{2x+8} (x + 2y)dydx$ = (after some easy computation) 2484/5.

4. [6 marks] Use double integrals and polar coordinates to find the area **in the first quadrant** between the lemniscate $r^2 = \cos 2\theta$ and the four-leaf rose $r = \cos 2\theta$.

First we sketch the region D in the first quadrant bounded by the two given curves:



A small analysis shows that we get this region as θ changes from 0 to $\frac{\pi}{4}$. The semiline that we show is there only to help us see that for a fixed angle θ the other coordinate r changes from the curve $r = \cos 2\theta$ (the curve on the boundary of the region D that is closer to the origin) to the curve $r = \sqrt{\cos 2\theta}$ (farther from the origin). Consequently, the area of D is $\int_0^{\pi/4} \int_{\cos(2\theta)}^{\sqrt{\cos(2\theta)}} rdr \;\; d\theta$. Easy computation yields $\int_0^{\pi/4} \int_{\cos(2\theta)}^{\sqrt{\cos(2\theta)}} rdr \;\; d\theta = \frac{1}{4} - \frac{\pi}{16}$.

1 mark for free (to celebrate the end of the term)