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Assignment 3 Brief Solutions

1. [4 marks].

(a) [2] Find $f_x(0,0)$ and find $f_y(x,y)$ if $f(x,y)=e^{xy}\sin(x+y+\pi)$.

(b) [2] Find all (four) second order partial derivatives of $g(x,y)=xy^2+\ln(x+y)$.

Solution.

[2] (a) $f_x(x,y)=ye^{xy}\sin(x+y+\pi)+e^{xy}\cos(x+y+\pi)$. So $f_x(0,0)=\cos(\pi)=-1$.

$f_y(x,y)=xe^{xy}\sin(x+y+\pi)+e^{xy}\cos(x+y+\pi)$.

[3] (b) $g_x(x,y)=y^2+\dfrac{1}{x+y}$, $g_y(x,y)=2xy+\dfrac{1}{x+y}$. So we have

$$g_{xx}(x,y)=-\frac{1}{(x+y)^2},\ g_{xy}(x,y)=2y-\frac{1}{(x+y)^2}$$

$$g_{yx}(x,y)=2y-\frac{1}{(x+y)^2} \text{ and } g_{yy}(x,y)=2x-\frac{1}{(x+y)^2}.$$

2. [5 marks]

(a) [1.5] Find the directional derivative of the function $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ in the direction of the unit vector $\mathbf{u}=\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$ at the point (1,-2).

(b) [2] Find the directions and the values of the smallest and the largest directional derivatives of the function $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ at the point (1,-2).

(c) [1.5] If $z^3-xy+yz+y^3-2=0$ defines z as a function on x and y, find $\dfrac{\partial z}{\partial x}$ at the point $(1,1,1)$.

Solutions. (a) $\nabla f(x,y)=(\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y})=\left(\dfrac{4xy^2}{(x^2+y^2)^2},-\dfrac{4yx^2}{(x^2+y^2)^2}\right)$ and $\nabla f(1,-2)=\left(\dfrac{16}{25},\dfrac{8}{25}\right)$.

So $D_u f(1,-2)=\nabla f(1,-2)\bullet u=\left(\dfrac{16}{25},\dfrac{8}{25}\right)\bullet\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)=\dfrac{16}{25}\dfrac{1}{\sqrt{2}}+\dfrac{8}{25}\dfrac{1}{\sqrt{2}}$.

(b) The largest value of the directional derivative at the given point is $|\nabla f(1,-2)|$, and that happens to be (after simplifying) $\dfrac{8}{25}\sqrt{5}$. The largest value is attained in the direction of the unit vector parallel to (and in the same direction as) $\nabla f(1,-2)$, which is (after simplifying) $\frac{1}{|\nabla f(1,-2)|}\nabla f(1,-2)=\frac{1}{\sqrt{5}}(2,1)$.

The smallest value of the directional derivative at the given point is $-|\nabla f(1,-2)|$, which is $-\frac{8}{25}\sqrt{5}$. The smallest value is attained in the direction of the unit vector $-\frac{1}{\sqrt{5}}(2,1)$.

(c) Recall that $\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$, where $F(x,y,z)=z^3-xy+yz+y^3-2$. We compute $\frac{\partial F}{\partial x}=-y$ and $\frac{\partial F}{\partial z}=3z^2+y$. Consequently $\frac{\partial z}{\partial x}=-\frac{-y}{3z^2+y}$, At the given point $y=1$ and $z=1$, so that, at that point $\frac{\partial z}{\partial x}=-\frac{1}{4}$.

3. [5 marks] First locate the local extrema of the function $g(x,y)=\frac{x+y}{x^2+y^2+8}$, and then use the second derivative test to classify these local extrema (as local minima, local maxima or neither).

Solution. $\frac{\partial g}{\partial x}=\frac{y^2-x^2+8-2xy}{\left(x^2+y^2+8\right)^2}$ and $\frac{\partial g}{\partial y}=\frac{x^2-y^2+8-2xy}{\left(x^2+y^2+8\right)^2}$. For critical points we solve $\frac{\partial g}{\partial x}=0=\frac{\partial g}{\partial y}$, which yields the system $y^2-x^2+8-2xy=0$, $x^2-y^2+8-2xy=0$. Add these two equations to get $16=4xy$ or $4=xy$. Put this in any of the original equations to get $x^2=y^2$ which means that $x=y$ or that $x=-y$. From this point it is easy to see that the solutions are (2,2) and (-2,-2).

We now compute $\frac{\partial^2 g}{\partial x^2}$, $\frac{\partial^2 g}{\partial y^2}$ and $\frac{\partial^2 g}{\partial x\partial y}$, and evaluate them at the two critical points.

For (-2,-2) we get $\left.\frac{\partial^2 g}{\partial x^2}\right|_{(-2,-2)}=\frac{1}{32}$, $\left.\frac{\partial^2 g}{\partial y^2}\right|_{(-2,-2)}=\frac{1}{32}$ and $\left.\frac{\partial^2 g}{\partial x\partial y}\right|_{(-2,-2)}=0$, so that $D=\frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial x^2}-\left(\frac{\partial^2 g}{\partial x\partial y}\right)^2$ is $\frac{1}{1024}$ at that point. Since D is positive and $\left.\frac{\partial^2 g}{\partial x^2}\right|_{(-2,-2)}$ is also positive, it follows that we have a local minimum at (-2,-2).

For (2,2) we get $\left.\frac{\partial^2 g}{\partial x^2}\right|_{(2,2)} = -\frac{1}{32}$, $\left.\frac{\partial^2 g}{\partial y^2}\right|_{(2,2)} = -\frac{1}{32}$ and $\left.\frac{\partial^2 g}{\partial x \partial y}\right|_{(2,2)} = 0$, so that $D = \frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial x^2} - \left(\frac{\partial^2 g}{\partial x \partial y}\right)^2$ is $\frac{1}{1024}$ at that point. Since D is positive and $\left.\frac{\partial^2 g}{\partial x^2}\right|_{(-2,-2)}$ is negative, it follows that we have a local maximum at (2,2).

4. [6 marks] Consider the function $f(x,y) = x^2 - x - y + y^2$ over the points in the closed disk bounded by the circle $x = 2\cos t$, $y = 2\sin t$. Find and classify the **absolute** extrema of the function $f(x,y)$ over the given domain.

Solution. Compute $\frac{\partial f}{\partial x} = 2x - 1$ and $\frac{\partial f}{\partial y} = 2y - 1$. Solving $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ yields $\left(\frac{1}{2}, \frac{1}{2}\right)$ as the only critical points. We now use the second derivative test: $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(1/2,1/2)} = 2$, $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(1/2,1/2)} = 2$ and $\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(1/2,1/2)} = 0$, so that $D = 4$. Since D>0 and $\frac{\partial^2 f}{\partial x^2} > 0$ at the critical point, it follows that the function has a local minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Now we see what happens over the boundary. The function *f* over the boundary circle reduces to
$f(x,y) = x^2 - x - y + y^2 = 4\cos^2 t + 2\cos t - 2\sin t + 4\sin^2 t = 2\cos t - 2\sin t + 4 = g(t)$ (a function on t, where t changes from 0 to 2π). Then $g'(t) = -2\sin t - 2\cos t$ and this is 0 for $t = \frac{3\pi}{4}$ or $t = \frac{7\pi}{4}$.

We now compute the values of f(x,y)=g(t) over these two points, as well as the value of f over the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ found above: $g\left(\frac{3\pi}{4}\right) = 4 - 2\sqrt{2}$, $g\left(\frac{7\pi}{4}\right) = 4 + 2\sqrt{2}$ and $f\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{2}$. Comparing these we see that we have an absolute maximum of $4 + 2\sqrt{2}$ happening when $t = \frac{7\pi}{4}$, that is, when $x = 2\cos\frac{7\pi}{4}$ and $y = 2\sin\frac{7\pi}{4}$, and the absolute minimum of $-\frac{1}{2}$ happening at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

5. [5 marks]. Find the dimensions of the rectangular box with no top and volume of 12 cubic meters which has the smallest surface area.

Solution. If the dimensions of the box are denoted by x, y and z (with z denoting the height), then we have that the volume is xyz and so $xyz = 12$. The surface area is

$S = xy + 2xz + 2yz$. From the fist equation we find that $z = \frac{12}{xy}$, so that $S(x,y) = xy + 2x\frac{12}{xy} + 2y\frac{12}{xy} = xy + \frac{24}{y} + \frac{24}{x}$. We want to find the absolute minimum of that function, with both x and y positive numbers. First we identify the critical points: $\frac{\partial S}{\partial x} = y - \frac{24}{x^2}$ and $\frac{\partial S}{\partial y} = x - \frac{24}{y^2}$ so that we need to solve the system $y - \frac{24}{x^2} = 0$ and $x - \frac{24}{y^2} = 0$. This is easy and the only solution is $(\sqrt[3]{24}, \sqrt[3]{24})$. The second partial derivatives are $\frac{\partial^2 S}{\partial x^2} = \frac{48}{x^3}$, $\frac{\partial^2 S}{\partial y \partial x} = 1$, $\frac{\partial^2 S}{\partial x \partial y} = 1$ and $\frac{\partial^2 S}{\partial y^2} = \frac{48}{y^3}$. At the critical point we compute $\frac{\partial^2 S}{\partial x^2} = 2$, $\frac{\partial^2 S}{\partial y \partial x} = 1$, $\frac{\partial^2 S}{\partial x \partial y} = 1$ and $\frac{\partial^2 S}{\partial y^2} = 2$, so that $D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$. Since $D > 0$ and since $\frac{\partial^2 S}{\partial x^2} > 0$ the critical point yields a local minimum. Since this is the only critical point and since the function S is differentiable over its domain, it follows that the critical point yields the absolute minimum of the function. The dimensions that yield the smallest surface area are $x\sqrt[3]{24}$ $y = \sqrt[3]{24}$ and $z = \frac{12}{\sqrt[3]{24}\sqrt[3]{24}}$.