136.272

Assignment 2 (Sections 14.3, 14.4, 15.1-15.2)

Posted: Oct.17 2005; handed Oct. 20, 2005. Due: Oct.24 2005 in class. (If you hand it in by Friday, Oct 21, you will get it back before the midterm.) Late assignments will not be accepted.

Show your work; providing answers without justifying them would not be sufficient.

1. [9 marks] A spiral curve is defined by the vector function $\vec{r}(t)=(4\cos t, 4\sin t, 3t)$.
   (a) Find the arc length function $s(t)$ measured from the point (4,0,0).
   (b) Reparametrize the curve in terms of the arc length function s measured from the point (4,0,0).
   (c) Compute the curvature of that spiral curve at any moment in terms of s.
   (d) Compute the curvature in terms of t directly from $\vec{r}(t)=(4\cos t, 4\sin t, 3t)$.
   (e) Find the equations of the normal and the osculating plane to the spiral at the point (4,0,0).

Solution.

[1.5] (a) $\vec{r}'(t)=(-4\sin t, 4\cos t, 3)$ and so $|\vec{r}'(t)|=\sqrt{16\sin^2 t+16\cos^2 t+9}=\sqrt{25}=5$. Consequently, $s(t)=\int_0^t 5du=5t$. Note that the lower limit of the integral is t=0 since at that moment we get the point (4,0,0).

[1.5] (b) If follows from (a) that $t=\frac{s}{5}$, so that $\vec{r}(s)=(4\cos\frac{s}{5}, 4\sin\frac{s}{5}, 3\frac{s}{5})$.

[2] (c) $\kappa(s)=\left|\frac{d\mathbf{T}}{ds}\right|$ where **T** is the unit tangent vector. So

$$\mathbf{T}(s)=\frac{\vec{r}'(s)}{|\vec{r}'(s)|}=\frac{\left(-\frac{4}{5}\sin\frac{s}{5},\frac{4}{5}\cos\frac{s}{5},\frac{3}{5}\right)}{1}=\left(-\frac{4}{5}\sin\frac{s}{5},\frac{4}{5}\cos\frac{s}{5},\frac{3}{5}\right).$$ So

$$\kappa(s)=\left|\frac{d\mathbf{T}}{ds}\right|=\left|\left(-\frac{4}{25}\cos\frac{s}{5},-\frac{4}{25}\sin\frac{s}{5},0\right)\right|=\frac{4}{25}.$$

[2] (d) $\kappa(t)=\left|\frac{\mathbf{T}'(t)}{\vec{\mathbf{r}}'(t)}\right|$. We have already computer that $|\vec{r}'(t)|=5$ and that $\vec{r}'(t)=(-4\sin t, 4\cos t, 3)$. So, $\mathbf{T}(t)=\frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}=\frac{(-4\sin t, 4\cos t, 3)}{5}$. So

$\mathbf{T}'(t)=\frac{(-4\cos t,-4\sin t,0)}{5}$. Consequently $|\mathbf{T}'(t)|=\frac{|(-4\cos t,-4\sin t,0)|}{5}=\frac{4}{5}$. Finally

$\kappa(t)=\left|\frac{\mathbf{T}'(t)}{\vec{\mathbf{r}}'(t)}\right|=\frac{4}{5}\frac{1}{5}=\frac{4}{25}$ as we have already found above.

[2] (e) $\vec{r}'(t) = (-4\sin t, 4\cos t, 3)$ is tangent to the curve and normal to the normal plane. At *t=0* (the moment we get the point (4,0,0)) we compute that $\vec{r}'(0) = (0,4,3)$. So, the equation of the normal plane through (4,0,0) is $0(x-4)+4(y-0)+3(z-0)=0$.

For the osculating plane we need the bi-normal vector, or, for that matter, any vector that is parallel to the bi-normal vector. The vector $\mathbf{T}(0)\times\mathbf{T}'(0)$ is such. Using what we have computed above we find that $\mathbf{T}(0)=\frac{1}{5}(0,4,3)$ and $\mathbf{T}'(0)=\frac{1}{5}(-4,0,0)$. The cross product of these two is $\frac{1}{25}(0,-12,16)$. So, the bi-normal plane has the equation $0(x-4)-\frac{12}{25}(y-0)+\frac{16}{25}(z-0)=0$.

2. [4 marks] Find the position vector $\mathbf{r}(t)$ of the particle with acceleration $\mathbf{a}(t)=(\sin t,\cos t,1)$, the initial velocity $\mathbf{v}(0)=(0,0,0)$ and initial position $\mathbf{r}(0)=(0,0,2)$. Where is the particle at the moment when $t=\pi$?

Solution. $\mathbf{v}(t)=\int \mathbf{a}(t)dt=\int(\sin t,\cos t,1)dt=(-\cos t+c_1,\sin t+c_2,t+c_3)$. Since $\mathbf{v}(0)=(0,0,0)$ we have that $(-\cos 0+c_1,\sin 0+c_2,0+c_3)=(0,0,0)$ which yields $c_1=1$, $c_2=0=c_3$, so that $\mathbf{v}(t)=(-\cos t+1,\sin t,t)$. Further

$\mathbf{r}(t)=\int \mathbf{v}(t)dt=\int(-\cos t+1,\sin t,t)dt=(-\sin t+t+d_1,-\cos t+d_2,\frac{t^2}{2}+d_3)$.

Since $\mathbf{r}(0)=(0,0,2)$, we have that $(-\sin 0+0+d_1,-\cos 0+d_2,\frac{0^2}{2}+d_3)=(0,0,2)$, from where we find that $d_1=0$, $d_2=1$ and $d_3=2$. So $\mathbf{r}(t)=(-\sin t+t,-\cos t+1,\frac{t^2}{2}+2)$. At $t=\pi$ we get $\mathbf{r}(\pi)=(-\sin\pi+\pi,-\cos\pi+1,\frac{\pi^2}{2}+2)=(\pi,2,\frac{\pi^2}{2}+2)$.

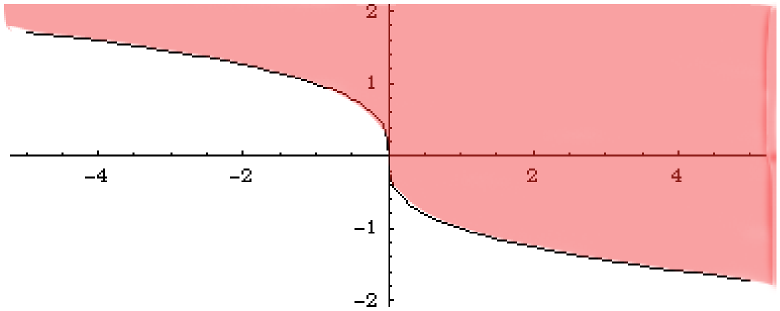
3. [6 marks] Determine **and sketch** (in the xy-plane) the domain of each of the following functions.

(a) $f(x,y)=\sqrt{x+y^3}$

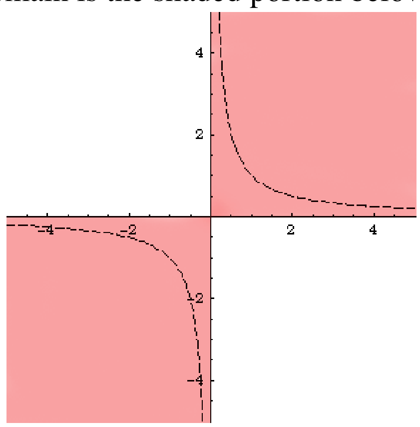
(b) $g(x,y)=\dfrac{x+y}{1-\sqrt{xy}}$

Solution.

[3] (a) We must have $x+y^3\ge 0$ or $y^3\ge -x$. The points satisfying that inequality are in the shaded region below.



[3] (b) $g(x,y)=\dfrac{x+y}{1-\sqrt{xy}}$ is well defined when $xy \ge 0$ and when $1-\sqrt{xy} \ne 0$. The first inequality happens for the points in the first and third quadrant (including the axes), while the second inequality happens for all points outside the curve $1-\sqrt{xy}=0$, that is, outside the hyperbola $xy=1$. The domain is the shaded portion below, excluding the dotted lines.



[6] 4. [6 marks]

(a) Find the limit or show it does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

(b) Find the limit or show it does not exist: $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2-1}{x^2+y^3}$

Solution.

[3] (a) Switch to polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0}\frac{1-\cos(r^2)}{r^2} \overset{L'Hospital}{=} \lim_{r\to 0}\frac{-2r\sin(r^2)}{2r} = 0$$

[3] (b) Along the curve $y=-x$ (passing through (1,-1)) we have

$$\lim_{\substack{(x,y)\to(1,-1)\\ \text{along that curve}}} \frac{x^2-1}{x^2+y^3} = \lim_{x\to 1}\frac{x^2-1}{x^2+(-x)^3} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x^2(1-x)} = \lim_{x\to 1}\frac{-(x+1)}{x^2} = -\frac{1}{2}.$$

On the other hand along the curve $y=-1$ (which also passes through (1,-1)), we have:

$$\lim_{\substack{(x,y)\to(1,-1)\\ \text{along } y=-1}} \frac{x^2-1}{x^2+y^3} = \lim_{x\to 1}\frac{x^2-1}{x^2+(-1)^3} = \lim_{x\to 1}\frac{x^2-1}{x^2-1} = \lim_{x\to 1} 1 = 1.$$

Since we got two different answers when approaching (1,-1) along two different curves, we conclude that the original limit does not exist.