THE UNIVERSITY OF MANITOBA

DATE: Oct. 26, 2005 MIDTERM EXAMINATION

DEPARTMENT & COURSE NO: 136.272 TIME: 1 hour

EXAMINATION: Calculus 3A EXAMINER: Dr. S.Kalajdzievski

NAME: (PRINT)____

STUDENT NUMBER:____________________________________

SIGNATURE:____
(I understand that cheating is a serious offense)

INSTRUCTIONS TO THE STUDENT

This is a one-hour exam. There are 3 pages of questions and one blank page for rough work. Check now that you have all 3 pages of questions.. Answer all the questions in the spaces provided. If necessary, you may continue your work on the reverse sides of the pages but PLEASE INDICATE CLEARLY that your work continues and where the continuation may be found. DO NOT detach any question pages from the exam.

The point value of each question is indicated to the left of the question number. The maximum score possible is 60 points.

Please present your work CLEARLY and LEGIBLY, and use a pen (not red) or a dark pencil. **Justify your answers unless otherwise stated.**

NO CALCULATORS, TEXTS, NOTES OR OTHER AIDS ARE PERMITTED.

Values

1. In all of (a), (b), (c) and (d) below, we consider the vector function $\mathbf{r}(t) = (\cos t + t\sin t, 0, \sin t - t\cos t)$ with $t \geq 0$.

[5] **(a)** Find the unit tangent vector $\mathbf{T}(t)$.
(b) Find the normal vector $\mathbf{N}(t)$.
(c) Compute the bi-normal vector $\mathbf{B}(t)$ at the point when $t = \pi$. Write the equation of the osculating plane at that point.
(d) Find the curvature at the point when $t = \pi$.

Solution.
(a) We compute $\mathbf{r}'(t) = (t\cos t, 0, t\sin t)$ and

$|\mathbf{r}'(t)| = \sqrt{t^2\cos^2 t + 0^2 + t^2\sin^2 t} = t$. So $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = (\cos t, 0, \sin t)$.

(b) $\mathbf{T}'(t) = (-\sin t, 0, \cos t)$ and $|\mathbf{T}'(t)| = 1$. So $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = (-\sin t, 0, \cos t)$.

(c) At the point $t = \pi$ we have $\mathbf{T}(\pi) = (\cos\pi, 0, \sin\pi) = (-1,0,0)$ and $\mathbf{N}(\pi) = (-\sin\pi, 0, \cos\pi) = (0,0,-1)$. So, $\mathbf{B}(\pi) = \mathbf{T}(\pi) \times \mathbf{N}(\pi) = (0,-1,0)$. This vector is normal to the osculating plane, and the plane passes through the point $\mathbf{r}(\pi) = (-1,0,\pi)$. So, its equation is $-y = 0$, i.e. $y = 0$ (which was obvious to start with, since the curve was given to be in the xz-plane).

(d) We have $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ and so $\kappa(\pi) = \dfrac{|\mathbf{r}'(\pi)\times\mathbf{r}''(\pi)|}{|\mathbf{r}'(\pi)|^3}$. Since $\mathbf{r}'(t) = (t\cos t, 0, t\sin t)$, we compute $\mathbf{r}'(\pi) = (-\pi, 0, 0)$. Since $\mathbf{r}''(t) = (\cos t - t\sin t, 0, \sin t + t\cos t)$, we find $\mathbf{r}''(t) = (-1, 0, -\pi)$. Further $\mathbf{r}'(\pi)\times\mathbf{r}''(\pi) = (0, -\pi^2, 0)$ and so $\kappa(\pi) = \dfrac{|\mathbf{r}'(\pi)\times\mathbf{r}''(\pi)|}{|\mathbf{r}'(\pi)|^3} = \dfrac{\pi^2}{\pi^3} = \dfrac{1}{\pi}$.

Alternative. $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1}{t}$ and so $\kappa(\pi) = \dfrac{1}{\pi}$.

2. It was observed that the velocity of an asteroid could be approximated by the function $\mathbf{v}(t) = (2t, 3t^2, 1-4t^3)$ (measured in millions of kilometers per day), where the time t is measured in days. The initial position of the asteroid (at $t = 0$ days) was at the point $(1,2,3)$ (measured in millions of kilometers). Find the position vector $\mathbf{r}(t)$. If the planet X is viewed as a point and if it is positioned at $(101, 1002, -9997)$ (in millions of kilometers) will the asteroid hit it? If yes, how many days after the initial observation?

Solution. $\mathbf{r}(t) = \int \mathbf{v}(t)dt = \int (2t, 3t^2, 1-4t^3)dt = (t^2 + c_1, t^3 + c_2, t - t^4 + c_3)$. Since $\mathbf{r}(0) = (1,2,3)$ we have $(0^2 + c_1, 0^3 + c_2, 0 - 0^4 + c_3) = (1,2,3)$, and so $\mathbf{r}(t) = (t^2+1, t^3+2, t - t^4 + 3)$. Since $\mathbf{r}(t) = (101, 1002, -9997)$ has no solution, the planet X will not be hit.

[6] 3. Show that the following limits do not exist.

(a) $$\lim_{(x,y)\to(0,0)} \frac{x^5 y}{x^6 + y^6}$$

Solution. Along $x = 0$ the limit is 0, while along $y = x$ it is $\dfrac{1}{2}$. That shows the limit does not exist.

[7] (b) $\lim_{(x,y)\to(0,0)} \frac{3xy^3}{x^2+y^6}$

Solution. Along $x=0$ the limit is 0. Along $x=y^3$ the limit is 3/2 and so the limit does not exist.

Values

[3] 4. **(a)** Use only the definition of $\lim_{(x,y)\to(a,b)} f(x,y)$ to prove that

$$\lim_{(x,y)\to(0,0)} 2+3\sqrt{x^2+y^2}=2$$

(b) Use polar coordinates to evaluate $\lim_{(x,y)\to(0,0)} \frac{3x^3}{x^2+y^2}$. Justify your steps.

Solution.

(a) We want to show that for every $\varepsilon>0$ there is some $\delta>0$ such that if $\sqrt{x^2+y^2}<\delta$ then $\left|2+3\sqrt{x^2+y^2}-2\right|<\varepsilon$. The latter equation is obviously equivalent to $3\sqrt{x^2+y^2}<\varepsilon$, i.e., to $\sqrt{x^2+y^2}<\varepsilon/3$. It is now obvious that choosing $\delta=\varepsilon/3$ will guarantee that $\sqrt{x^2+y^2}<\delta$ implies $\left|2+3\sqrt{x^2+y^2}-2\right|<\varepsilon$.

(b) The given limit is the same as to $\lim_{r\to 0}\frac{3r^3\cos^3\theta}{r^2}$ in polar coordinates. Simplifying a bit we get $3\lim_{r\to 0} r\cos^3\theta$. Now $-r\le r\cos^3\theta\le r$ since $-1\le\cos^3\theta\le 1$. It follows by the squeeze theorem that $\lim_{r\to 0} r\cos^3\theta=0$ and so $3\lim_{r\to 0} r\cos^3\theta=0$.

(Note: I do not take out marks if you did not do it "my way", as long as what you did is correct and has no substantial gaps. For example, just saying that $3\lim_{r\to 0} r\cos^3\theta=0$ is a bad argument, for if applied to, say, $3\lim_{r\to 0}\frac{r}{\sin\theta}$ would imply that that limit is 0, which it is NOT (why?).