MATH 2720 Multivariable Calculus
TEST 2 SOLUTIONS
March 11, 2009
(5:30-6:30, 205 Armes)

NAME: ____________________ **Student number:** _______

(If you need more space use the backside and indicate that you have done so.)

[9] **1.** Consider the function $z = x^2(1+y^2)$.

(a) Find the slope of the line passing through the point (1,1,2) and tangent to the curve of intersection of the surface $z = x^2(1+y^2)$ and the plane $y = 1$.

(b) Evaluate $\dfrac{\partial^2 z}{\partial x \partial x}$ and $\dfrac{\partial^2 z}{\partial y \partial x}$.

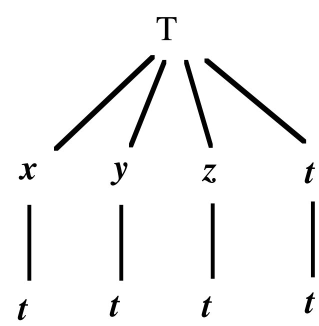
Solution. **(a)** The slope of that line is the same as $\dfrac{\partial z}{\partial x}(1,1)$. We compute:

$\dfrac{\partial z}{\partial x}(x,y) = 2x(1+y^2)$, so that $\dfrac{\partial z}{\partial x}(1,1) = 4$.

(b) $\dfrac{\partial^2 z}{\partial x \partial x}(x,y) = 2(1+y^2)$; $\dfrac{\partial^2 z}{\partial y \partial x} = 4xy$.

[11] **2.** A weather balloon moves along the curve $x = t,\ y = 2t,\ z = t - t^2$, where t stands for the elapsed time measured in hours (and x, y and z are the coordinates of the balloon). The thermometer attached to the balloon gives the temperature of $T(x,y,z,t) = \dfrac{xy}{1+z}(1+t)$ (in degrees Celsius). Find the rate of change of the temperature at the time when $t = 1$.

Solution. Here is the tree diagram in this case:



The functions x, y, and z are given in the statement of the problem, while the last vertical line in the tree-diagram stands for the trivial function t on t ($t = t$). We now differentiate with respect to the bottom variable t:

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t}\frac{dt}{dt} =$$

$$= \frac{y}{1+z}(1+t) + \frac{x}{1+z}(1+t)2 - \frac{xy}{(1+z)^2}(1+t)(1-2t) + \frac{xy}{1+z}$$

Now, when $t = 1$ we compute from $x = t,\ y = 2t,\ z = t - t^2$ that $x = 1,\ y = 2,\ z = 0$. We substitute that in the above expression for $\dfrac{dT}{dt}$ to get that at that moment

$$\frac{dT}{dt} = \frac{2}{1}(1+1) + 2\frac{1}{1}(1+1) - \frac{2}{1}(1+1)(1-2) + \frac{2}{1} = 4+4+4+2 = 14 \,{}^{\text{degrees}}\!/\!_{\text{hour}}.$$

[11] **3.** Consider the function $f(x,y) = y^2 e^{2x}$.

(a) Find the directional derivative of this function at the point $P(0,1)$ and in the direction of the unit vector $\mathbf{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

(b) Find the unit vector in the direction in which *f* increases most rapidly at *P* and give the rate of change in that direction. Find the unit vector in the direction in which *f* decreases most rapidly at *P* and give the rate of change in that direction.

Solution. We compute the gradient of *f*: $\nabla f = (2y^2 e^{2x}, 2ye^{2x})$. At the given point *P* we have $\nabla f(0,1) = (2,2)$.

(a) $\mathbf{D}_{\mathbf{u}} f(0,1) = \nabla f(0,1) \bullet \mathbf{u} = (2,2) \bullet (\frac{1}{2}, \frac{\sqrt{3}}{2}) = 1 + \sqrt{3}$.

(b) The function *f* increases most rapidly in the direction of the gradient vector $\nabla f(0,1) = (2,2)$. The unit vector in that direction is $\frac{\nabla f(0,1)}{|\nabla f(0,1)|} = \frac{(2,2)}{\sqrt{8}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The rate of change in that direction is $|\nabla f(0,1)| = \sqrt{8}$. The direction in which *f* decreases most rapidly is $-\nabla f(0,1) = -(2,2)$. The associated unit vector is $-(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and the rate of change in that direction is $-|\nabla f(0,1)| = -\sqrt{8}$.

[12] **4.** Find the equation of the tangent plane to the surface defined by $xy + yz + xz = 11$ at the point $(1,2,3)$.

Solution. The gradient vector of $F(x,y,z) = xy + yz + xz$ at the given point is normal to the tangent plane. We compute: $\nabla F(x,y,z) = (y+z, x+z, x+y)$, and $\nabla F(1,2,3) = (5,4,3)$. So the equation of the tangent plane is:
$5(x-1) + 4(y-2) + 3(z-3) = 0$.

[11] **5.** Find the point in the plane $x - y + z = 1$ that is closest to the point $(-1,1,2)$. Justify your answer by using the second (partial) derivative test.

Solution. We are minimizing the distance between a point (x,y,z) on the plane and the point $(-1,1,2)$. The formula is: $d = \sqrt{(x+1)^2 + (y-1)^2 + (z-2)^2}$. As explained in class, minimizing *d* is the same as minimizing $d^2 = (x+1)^2 + (y-1)^2 + (z-2)^2$. Since the point (x,y,z) is on the plane it satisfies the equation of the plane, so that $z = 1 - x + y$. Hence we are minimizing
$d^2 = (x+1)^2 + (y-1)^2 + (1-x+y-2)^2 = (x+1)^2 + (y-1)^2 + (-1-x+y)^2$. We call this expression $f(x,y)$. First we find the critical points, by solving $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 2(x+1) - 2(-1-x+y) = 4x - 2y + 4 .$$

$$\frac{\partial f}{\partial y} = 2(y-1) + 2(-1-x+y) = -2x + 4y - 4 .$$

Solving $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ gives $x = -\frac{2}{3}$ and $y = \frac{2}{3}$.
To see what kind of point this is we use the second derivative test.
$f_{xx} = 4$, $f_{xy} = -2$, $f_{yx} = -2$, $f_{yy} = 4$. We then compute $D = 16 - 4 = 12$, so that $D > 0$. Since $f_{xx} = 4 > 0$ we have a local minimum. It follows from the nature of the problem that this is the absolute minimum of the function.