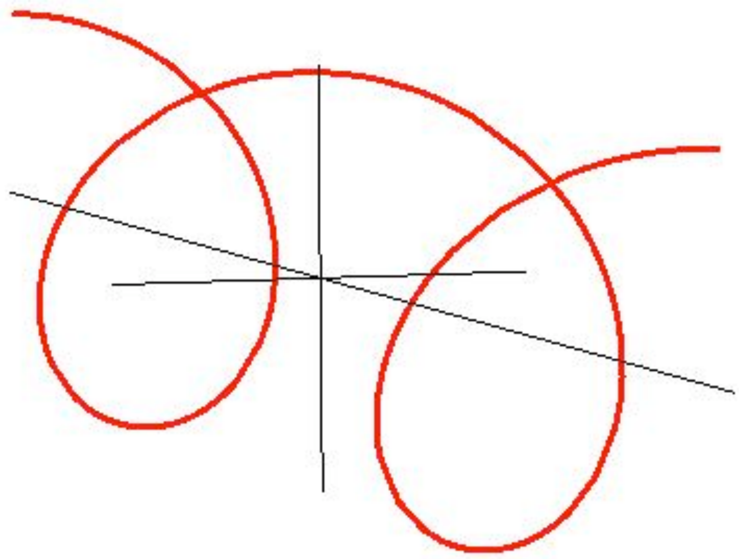
MATH 2720 Multivariable Calculus SOLUTIONS
Midterm Exam
February 11, 2009
(5:30-6:30, 209 Armes)

(If you need more space use the back side and indicate that you have done so.)

1. We are given the vector-valued function $\mathbf{r}(t) = (t, \sin 3t, \cos 3t)$

(a) Sketch this curve.

Solution.



(b) Find the parametric equations of the tangent line of the curve $\mathbf{r}(t)$ at the point $(\frac{\pi}{3}, 0, -1)$.

Solution. $\mathbf{r}'(t) = (1, 3\cos 3t, -3\sin 3t)$, and at the given point (when $t = \frac{\pi}{3}$), we have $\mathbf{r}'(t) = (1, -3, 0)$. So, the parametric equations of the tangent line are $x = t + \frac{\pi}{3}$, $y = -3t$, $z = -1$.

(c) Find the arc-length of that curve for $0 \le t \le 5$.

Solution. $|\mathbf{r}'(t)| = |(1, 3\cos 3t, -3\sin 3t)| = \sqrt{1+9} = \sqrt{10}$. So, the arc length we want is $\int_0^5 |\mathbf{r}'(t)|\,dt = \int_0^5 \sqrt{10}\,dt = \sqrt{10}t \Big|_0^5 = 5\sqrt{10}$.

2. Consider $\mathbf{r}(t) = (\cos 2t, \sin 2t, t)$, $t > 0$.

(a) Find the unit normal vector $\mathbf{T}(\frac{\pi}{2})$, the normal vector $\mathbf{N}(\frac{\pi}{2})$ and the binormal vector $\mathbf{B}(\frac{\pi}{2})$.

(b) Write an equation of the osculating plane of the curve $\mathbf{r}(t) = (\cos 2t, \sin 2t, t)$ at the point $(-1, 0, \frac{\pi}{2})$.

Solution. **(a)** $\mathbf{r}'(t) = (-2\sin 2t, 2\cos 2t, 1)$ and

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{4+1}}(-2\sin 2t, 2\cos 2t, 1) = \dfrac{1}{\sqrt{5}}(-2\sin 2t, 2\cos 2t, 1)$, so that

$\mathbf{T}(\frac{\pi}{2}) = \dfrac{1}{\sqrt{5}}(0,-2,1)$. Further $\mathbf{T}'(t) = \dfrac{1}{\sqrt{5}}(-4\cos 2t, -4\sin 2t, 0)$, so that

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\sqrt{5}}{4}\dfrac{1}{\sqrt{5}}(-4\cos 2t, -4\sin 2t, 0) = \dfrac{1}{4}(-4\cos 2t, -4\sin 2t, 0)$. Consequently

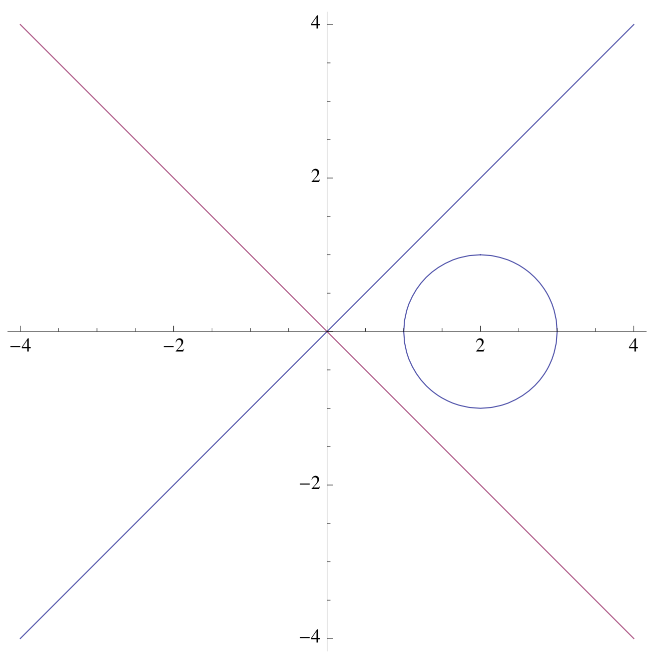
$\mathbf{N}(\frac{\pi}{2}) = \dfrac{1}{4}(4,0,0) = (1,0,0)$. Now we find that

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{5}}(0,-2,1) \times (1,0,0) = \frac{1}{\sqrt{5}}(0,1,2)$$

(b) The osculating plane is perpendicular to $\mathbf{B}\left(\frac{\pi}{2}\right)$ and passes through $(-1,0,\frac{\pi}{2})$.

So, its equation is $0 + \dfrac{1}{\sqrt{5}}(y-0) + \dfrac{2}{\sqrt{5}}(z - \dfrac{\pi}{2}) = 0$.

3. Sketch the domain of the function $f(x,y) = \dfrac{xy}{(x-y)(x+y)}$. Is this function continuous at the points of the circle $(x-2)^2 + y^2 = 1$? Why?



Solution. The only restriction for *f* is that the denominator should not be 0. This gives $x \neq y$, and $-x \neq y$. So, the domain consists of all of the points out of the two lines shown in the picture below. In the illustration we also show the circle $(x-2)^2 + y^2 = 1$: it is obvious that the circle is in the domain of the function (since it does not have any points in common with the two lines). Since rational functions are continuous over all of its domain, it follows that this function is continuous over the circle.

4.

(a) Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy + y^3}{x^2 + y^2}$ does not exist.

(b) Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^3 y}{x^6 + y^2}$ does not exist.

(c) Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2}{\sqrt{x^2 + y^2}}$ exists. Evaluate it! [Hint: switch to polar coordinates and use the Squeeze theorem.]

Solution. **(a)** Along $y = mx$ we get

$\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx}} \frac{xy+y^3}{x^2+y^2} = \lim\limits_{x\to 0} \frac{kx^2+k^3x^3}{x^2+k^2x^2} = \lim\limits_{x\to 0} \frac{k+xk^3}{1+k^2} = \frac{k}{1+k^2}$, and since this value depends on k, it follows that the limit in the question does not exist.

(b) Along the line $x = 0$ we get $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} \frac{x^3y}{x^6+y^2} = \lim\limits_{y\to 0} \frac{0}{0^6+y^2} = 0$. Along $y = x^3$ we get $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=x^3}} \frac{x^3y}{x^6+y^2} = \lim\limits_{y\to 0} \frac{x^6}{x^6+x^6} = \frac{1}{2}$. Since these two values (0 and $\frac{1}{2}$ are distinct, the limit in the question does not exist.

(c) Switching to polar gives: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = \lim\limits_{r\to 0} \frac{r^2\cos^2\theta}{r} = \lim\limits_{r\to 0} r\cos^2\theta$. Now $0 \le \cos^2\theta \le 1$, so that $0 \le r\cos^2\theta \le r$. The squeeze theorem then gives $\lim\limits_{r\to 0} 0 \le \lim\limits_{r\to 0} r\cos^2\theta \le \lim\limits_{r\to 0} r$, so that $0 \le \lim\limits_{r\to 0} r\cos^2\theta \le 0$, and thus $\lim\limits_{r\to 0} r\cos^2\theta = 0$.

5. **(a)** State the definition of $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ That is, what does it mean to say that $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to some number *L*?

Solution. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|(x,y)-(a,b)| < \delta$ then $|f(x,y)-L| < \varepsilon$.

(b) Show **using the definition of *limit*** that $\lim\limits_{(x,y)\to(0,0)} \sqrt[3]{8x^2+8y^2} = 0$. No points will be given if other methods are used.

Solution. Start with any $\varepsilon > 0$. We want to find $\delta > 0$ such that if

(1) $$|(x,y)-(0,0)| < \delta$$

then

(2) $$\left|\sqrt[3]{8x^2+8y^2} - 0\right| < \varepsilon$$

(1) simplifies to $\sqrt{x^2+y^2} < \delta$. We now focus on (2) and simplify it:

$(2) \Leftrightarrow \left|\sqrt[3]{8x^2+8y^2}\right| < \varepsilon \Leftrightarrow 2\sqrt[3]{x^2+y^2} < \varepsilon \Leftrightarrow \sqrt[3]{x^2+y^2} < \frac{\varepsilon}{2}$. Further, this can be modified as follows (recall that our goal is to relate (1) and (2)):

$\sqrt[3]{x^2+y^2} < \frac{\varepsilon}{2} \Leftrightarrow \left(\sqrt[3]{x^2+y^2}\right)^3 < \frac{\varepsilon^3}{8} \Leftrightarrow x^2+y^2 < \frac{\varepsilon^3}{8} \Leftrightarrow \sqrt{x^2+y^2} < \sqrt{\frac{\varepsilon^3}{8}}$. Summarizing, (2) can be written as $\sqrt{x^2+y^2} < \sqrt{\frac{\varepsilon^3}{8}}$. It is now obvious that if we take $\delta = \sqrt{\frac{\varepsilon^3}{8}}$, then (1) implies (2) as wanted.