136.271
Midterm Exam 1
SOLUTIONS
February 27 2003

(60 minutes; **justify** your answers unless otherwise stated; no calculators)

Note: the marks for the questions add up to 65, for 32.5% of your mark: the extra 2.5% are bonus.

1. [13]

(a) Finish off the following definition: A sequence $\{a_n\}$ converges to a number L (written $\lim_{n\to\infty} a_n = L$) if ....

(b) Use the (above) definition of a convergent sequence to show that $\lim_{n\to\infty}\left(\frac{2n-1}{n+1}-1\right)=1$.

Solution.

(a) A sequence $\{a_n\}$ converges to a number L (written $\lim_{n\to\infty} a_n = L$) if for every $\varepsilon > 0$, there is a number *N*, such that if *n*>*N* then $|a_n - L| < \varepsilon$.

(b) Take any $\varepsilon > 0$. Want to find *N* such that if *n*>*N* then $|a_n - L| < \varepsilon$. The last inequality in case of the sequence in this problem becomes $\left|\frac{2n-1}{n+1}-1-1\right| < \varepsilon$, which simplifies to $\left|\frac{-3}{n+1}\right| < \varepsilon$, which in turn simplifies to $\frac{3}{n+1} < \varepsilon$, which in turn means $\frac{3}{\varepsilon} < n+1$, or, finally $\frac{3}{\varepsilon} - 1 < n$. So we want to find a number *N*, such that if *n*>*N* then $\frac{3}{\varepsilon} - 1 < n$. Choose *N* to be any number larger than $\frac{3}{\varepsilon} - 1$ (that is, $N > \frac{3}{\varepsilon} - 1$). Then, if *n*>*N* then, $\frac{3}{\varepsilon} - 1 < n$, and so $n > \frac{3}{\varepsilon} - 1$.

2. [22] All of the series in this question are positive. Determine if the given series converges or diverges by using any appropriate test.

(a) $\sum_{n=2}^{\infty} \frac{4^n (n+1)}{2^{2n}(2n-1)}$

(b) $\sum_{n=2}^{\infty} \frac{\sqrt{n+1}}{n^2-1}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

Solution.

(a) Note that $\frac{4^n(n+1)}{2^{2n}(2n-1)} = \frac{4^n(n+1)}{4^n(2n-1)} = \frac{(n+1)}{(2n-1)}$ and that $\lim_{n\to\infty}\left(\frac{n+1}{2n-1}\right) = \frac{1}{2}$. It follows from the divergence test that the series diverges.

(b) We use the limit comparison test with $\sum_{n=2}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ (which, as we know, converges).

$$\lim_{n\to\infty}\left(\frac{\frac{\sqrt{n+1}}{n^2-1}}{\frac{1}{n^{\frac{3}{2}}}}\right) = \lim_{n\to\infty}\left(\frac{n^{\frac{3}{2}}\sqrt{n+1}}{n^2-1}\right) = \lim_{n\to\infty}\left(\frac{n^{\frac{3}{2}}\sqrt{n}\sqrt{1+\frac{1}{n}}}{n^2(1-\frac{1}{n^2})}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{1+\frac{1}{n}}}{(1-\frac{1}{n^2})}\right) = 1.$$

Since we got a finite number both series converge/diverge simultaneously. So, since $\sum_{n=2}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges, it follows that $\sum_{n=2}^{\infty}\frac{\sqrt{n+1}}{n^2-1}$ converges too.

(c) Consider the function $f(x) = \frac{1}{x\ln x}$, $x \geq 2$. Notice that $f(n)$, $n \geq 2$, yields the general term of the series. Also notice that $f(x)$ is continuous, positive and decreasing (all of these are obvious). So, we can use the integral test.

$$\int_2^{\infty} f(x)dx = \int_2^{\infty}\frac{1}{x\ln x}dx = \lim_{a\to\infty}\int_2^{a}\frac{1}{x\ln x}dx \overset{u=\ln x}{=} \lim_{a\to\infty}\int_{x=2}^{x=a}\frac{1}{u}du = \lim_{a\to\infty}\ln u\Big|_{x=2}^{x=a}$$

.

$$= \lim_{a\to\infty}(\ln\ln x)\Big|_2^a = \lim_{a\to\infty}(\ln\ln a - \ln\ln 2) = \infty$$

So, the series diverges.

3.[17] Find the radius of convergence and the interval of convergence of the series

(a) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

(b) $\sum_{n=4}^{\infty}\frac{5^n(x-3)^n}{n-3}$

Solution.

(a) We use the ration test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{|x^{n+1}|}{(n+1)!}}{\frac{|x^n|}{n!}} = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0$ and since this is always less than 1 it follows that the series converges for all numbers x. So, the radius of convergence is infinity and the interval of convergence is $(-\infty,+\infty)$.

(b) Ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{5^{n+1}\left|(x-3)^{n+1}\right|}{(n-2)}}{\frac{5^n\left|(x-3)^n\right|}{(n-3)}} = 5|x-3|\lim_{n\to\infty}\frac{n-3}{n-2} = 5|x-3|$. So, the series converges when $5|x-3|<1$. We solve this:

$5|x-3|<1 \Leftrightarrow |x-3|<\frac{1}{5} \Leftrightarrow -\frac{1}{5}<x-3<\frac{1}{5} \Leftrightarrow -\frac{1}{5}+3<x<\frac{1}{5}+3 \Leftrightarrow \frac{14}{5}<x<\frac{16}{5}$, where the symbol $\Leftrightarrow$ should be read as "means the same as". We conclude that the radius of convergence is $\frac{1}{5}$. To establish the interval of convergence we need to take a look at the cases when $x=\frac{14}{5}$ and $x=\frac{16}{5}$.

(i) $x=\frac{14}{5}$. In this case the series becomes $\sum_{n=4}^{\infty}\frac{5^n(-\frac{1}{5})^n}{n-3} = \sum_{n=4}^{\infty}\frac{(-1)^n}{n-3}$ which converges by the alternating series test.

(ii) $x=\frac{16}{5}$. Now we have $\sum_{n=4}^{\infty}\frac{5^n(\frac{1}{5})^n}{n-3} = \sum_{n=4}^{\infty}\frac{1}{n-3}$ which diverges (by the limit comparison test with $\sum_{n=4}^{\infty}\frac{1}{n}$, say).

So, the interval of convergence is $[\frac{14}{5},\frac{16}{5})$.

4. [13] **(a)** Find the interval of convergence and the sum of (the closed form expression for) the series

$$\sum_{n=0}^{\infty}5x^{3n} = 5+5x^3+5x^6+5x^9+\ldots.$$

(b) Use (a) to compute the sum of the series $5-\frac{5}{8}+\frac{5}{64}-\frac{5}{512}+\ldots.$

Solution.

(a) For the interval of convergence we use the ration test again:

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5|x|^{3n+3}}{5|x|^{3n}} = |x|^3$. We solve $|x|^3<1$, to get $-1<x<1$. It is visible that the series diverges when $x=-1$ or $x=1$. So, the interval of convergence is $(-1,1)$.

Denote $\sum_{n=0}^{\infty}x^n = f(x)$. Observe that $\sum_{n=0}^{\infty}5x^{3n} = 5\sum_{n=0}^{\infty}x^{3n} = 5\sum_{n=0}^{\infty}\left(x^3\right)^n$ and since, obviously, $\sum_{n=0}^{\infty}\left(x^3\right)^n = f(x^3)$, it follows that $5\sum_{n=0}^{\infty}\left(x^3\right)^n = 5f(x^3)$ and so $\sum_{n=0}^{\infty}5x^{3n} = 5f(x^3)$. Since (geometric series) $\sum_{n=0}^{\infty}x^n = f(x) = \frac{1}{1-x}$, it follows that $5f(x^3) = 5\frac{1}{1-x^3}$. All t his is true over the common interval of convergence $(-1,1)$. So, summarizing, $\sum_{n=0}^{\infty} 5x^{3n} = 5f(x^3) = \frac{5}{1-x^3}$, for x in $(-1,1)$.

(b) $5 - \frac{5}{8} + \frac{5}{64} - \frac{5}{512} + ...$ is just the value of the series in (a) when $x = -\frac{1}{2}$. So, by (a), it is equal to $\frac{5}{1-\left(-\frac{1}{2}\right)^3}$.