136.271

Assignment 4: Section 9.8 and Uniform Convergence

(Due April 7 in class)

1.

(a) Use the binomial series to expand $x(1-x)^{-2}$. Simplify your answer.

(b) Use part (a) to find the sum of the series $\sum_{n=1}^{\infty}\frac{n}{2^n}$. (No marks if other methods are used.)

Solution. (a) According to the Binomial theorem, we have

$(1-x)^{-2}=1+\sum_{n=1}^{\infty}\frac{(-2)(-3)...(-2+n-1)}{n!}(-x)^n$. We simplify a bit

$$(1-x)^{-2}=1+\sum_{n=1}^{\infty}\frac{(-2)(-3)...(-2-n+1)}{n!}(-x)^n=1+\sum_{n=1}^{\infty}\frac{(-1)^n(2)(3)...(n+1)}{n!}(-1)^n x^n=$$

$$1+\sum_{n=1}^{\infty}\frac{(-1)^n(2)(3)...(n+1)}{n!}(-1)^n x^n=1+\sum_{n=1}^{\infty}(n+1)x^n=\sum_{n=0}^{\infty}(n+1)x^n$$

So, $x(1-x)^{-2}=x\sum_{n=0}^{\infty}(n+1)x^n=\sum_{n=0}^{\infty}(n+1)x^{n+1}=\sum_{n=1}^{\infty}nx^n$.

(b) We observe that $\sum_{n=1}^{\infty}\frac{n}{2^n}$ is what we get if we put $x=\frac{1}{2}$ in $\sum_{n=1}^{\infty}nx^n$. So, we get the same value if we substitute $x=\frac{1}{2}$ in $x(1-x)^{-2}$, which is $\frac{1}{2}(1-\frac{1}{2})^{-2}=2$.

2. Given the sequence of functions $\{f_n(x)\}$ find the pointwise limit $f(x)$ and then show that the sequence $\{f_n(x)\}$ converges uniformly to $f(x)$.

(a) $f_n(x)=\frac{n+x}{n}$ over the interval [0,1].

(b) $f_n(x)=\frac{\ln(1+nx)}{n}$ over the interval [1,2].

Solution.

(a) First we find the pointwise limit: $\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{n+x}{n}=\lim_{n\to\infty}1+\frac{x}{n}=1$. So, $f(x)=1$ for every x in [0,1]. Now we take a look at $\lim_{n\to\infty}\max_{x\ in\ [0,1]}|f_n(x)-f(x)|$. We find that $|f_n(x)-f(x)|=\left|\frac{n+x}{n}-1\right|=\left|\frac{n+x-n}{n}\right|=\left|\frac{x}{n}\right|=\frac{x}{n}$. This function is obviously increasing over the interval [0,1], so its maximal value is attained at x=1; so

$\max_{x \text{ in } [0,1]} |f_n(x) - f(x)| = \max_{x \text{ in } [0,1]} \frac{x}{n} = \frac{1}{n}$**.** Obviously, $\lim_{n\to\infty} \max_{x \text{ in } [0,1]} |f_n(x) - f(x)| = \lim_{n\to\infty} \frac{1}{n} = 0$. So, the convergence is indeed uniform.

(b) We follow the above procedure: $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\ln(1+nx)}{n} = \lim_{n\to\infty} \frac{\frac{x}{1+nx}}{1} = 0$**;** we have used the L'Hospital's rule in the middle equality (we have differentiated the denominator and numerator separately **with respect to n**. So, $f(x) = 0$ and we have the pointwise limit.

For the rest we again consider $\lim_{n\to\infty} \max_{x \text{ in } [1,2]} |f_n(x) - f(x)|$. This time we have $\left|\frac{\ln(1+nx)}{n} - 0\right| = \frac{\ln(1+nx)}{n}$. We find the maximum of this function for x in $[1,2]$ (we differentiate with respect to x): $\left(\frac{\ln(1+nx)}{n}\right)' = \frac{1}{n}\left(\frac{1}{1+nx}\right)n = \frac{1}{1+nx}$. This is obviously always larger than 0, so that the function $\frac{\ln(1+nx)}{n}$ increases over $[1,2]$, and so its maximal value is attained at x=2. We compute $\left.\frac{\ln(1+nx)}{n}\right|_{x=2} = \frac{\ln(1+2n)}{n}$**.** Now we find the limit (use L'Hospital's rule again): $\lim_{n\to\infty} \frac{\ln(1+2n)}{n} = \lim_{n\to\infty} \frac{2}{1+2n} = 0$ and so the convergence is uniform.

3. Given the sequence of functions $\{f_n(x)\}$ find the pointwise limit $f(x)$ and then show that the sequence $\{f_n(x)\}$ does **NOT** converge uniformly to $f(x)$.

(a) $f_n(x) = \frac{n+x}{n}$ over the interval $[0,\infty)$

(b) $f_n(x) = \frac{n}{e^{nx^2}}$ over the interval $[0,1]$.

Solution.

(a) Follow the solution of 2(a) to the point when we found that $|f_n(x) - f(x)| = \frac{x}{n}$**.** Our goal now is to show that $\max_{x \text{ in } [0,\infty]} |f_n(x) - f(x)| = \max_{x \text{ in } [0,\infty]} \frac{x}{n}$ does not tend to 0 as n tends to infinity. (Note the difference with respect to 2(a): the domain in this question is $[0,\infty)$, not $[0,1]$ as it was in 2(a)). Choose x=n (this value is certainly in $[0,\infty)$. For this choice of x we have that $\frac{x}{n}$ is $\frac{n}{n} = 1$. This obviously tends to 1 as n tends to infinity (in fact, it is always 1). Consequently, the maximum $\max_{x \text{ in } [0,\infty]} \frac{x}{n}$, which is certainly at least as much as what we get when x=n, could not approach to 0 as n approaches infinity. So, the convergence is NOT uniform.

(b) First the pointwise limit: $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{e^{nx^2}} = \lim_{n\to\infty} \frac{1}{x^2 e^{nx^2}} = 0$ (we have used L'Hospital again – the differentiation was done **with respect to n.)** So, $f(x) = 0$. We find that $\left|f_n(x) - f(x)\right| = \left|\frac{n}{e^{nx^2}} - 0\right| = \frac{n}{e^{nx^2}}$, so that we look for the maximal value of that function over the interval [0,1].We differentiate with respect to x, to get $\left(\frac{n}{e^{nx^2}}\right)' = \frac{n(-2nx)}{e^{nx^2}} = \frac{-2n^2x}{e^{nx^2}}$**.** This is always negative, so that the function is decreasing over [0,1]. Consequently, its maximum is attained at x=0. We compute the value: $\left.\frac{n}{e^{nx^2}}\right|_{x=0} = n$ and the limit of this is certainly not 0 as n tends to infinity. So, the convergence is not uniform, as claimed.