136.271

Assignment 3: Solutions

1. [6 marks] Find the sums of the following series.

(a) $\sum_{n=2}^{\infty} n(n-1)x^n$, $|x|<1$.

(b) $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$.

Solution.

(a) $\sum_{n=2}^{\infty} n(n-1)x^n = x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\sum_{n=2}^{\infty}\left(x^n\right)'' = x^2\sum_{n=0}^{\infty}\left(x^n\right)''$, where the last equality is true because the first two terms are annihilated by differentiation anyway. Continuing, we have $x^2\sum_{n=0}^{\infty}\left(x^n\right)'' = x^2\left(\sum_{n=0}^{\infty} x^n\right)'' = x^2\left(\frac{1}{1-x}\right)'' = 2x^2\left(\frac{1}{(1-x)^3}\right)$.

(b) $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$ is clearly the value of the function in (a) when $x=\frac{1}{2}$, and so, by (a), it is equal to $2\left(\frac{1}{2}\right)^2\left(\frac{1}{(1-\left(\frac{1}{2}\right))^3}\right)$

2. [5 marks] Find the Maclaurin series representation of the following functions.

(a) e^{3x}

(b) $\sin^2 x$

Solution.

(a) Since $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, it follows that $e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$.

(b) $\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x$ is an old identity. We know that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, so that $\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$. Consequently $\frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$ and so $\sin^2 x = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$.

3. [6 marks] Find the sum of the series.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

(b) $\sum_{n=2}^{\infty} \frac{x^{3n+1}}{n!}$

(c) $\sum_{n=0}^{\infty} \frac{x^n}{2^n (n+1)!}$

Solution.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(b) $\sum_{n=2}^{\infty} \frac{x^{3n+1}}{n!} = x\sum_{n=2}^{\infty} \frac{x^{3n}}{n!} = x\left(\sum_{n=0}^{\infty} \frac{\left(x^3\right)^n}{n!} - 1 - x^3\right) = x(e^{x^3} - 1 - x^3).$

(c) Note first that if x=0 then the value of the series is 1. For $x \neq 0$ we do as follows:

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n (n+1)!} = \frac{2}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}(n+1)!} = \frac{2}{x}\sum_{n=1}^{\infty} \frac{x^n}{2^n n!} = \frac{2}{x}\left(\sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^n}{n!} - 1\right) = \frac{2}{x}(e^{\frac{x}{2}} - 1).$$

4. [4 marks] Evaluate the following integrals as power series.

(a) $\int_0^x \sin(t^2)dt$

(b) $\int_0^x e^{t^3} dt$

Solution.

(a) Since $\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$ for every *t*, we have that $\sin(t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!}$. So

$$\int_0^x \sin(t^2)dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!}\right)dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$$

(b) $\int_0^x e^{t^3} dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{t^{3n}}{n!}\right)dt = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)n!}.$

5. [4 marks] Show that the Lagrange remainder in the Taylor's formula for the following functions tends to 0 as n tends to infinity, thus establishing that the functions are equal to their power series representations.

(a) $\cos 4x$

(b) e^{-2x}

Solution.

(a) $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ for some z between c and x. Since here we have that $f(x) = \cos 4x$, the n-th derivative of this function is some of $4^n \sin x,\ 4^n \cos x,\ -4^n \sin x$ and $-4^n \cos x$, so that in all cases $|f(x)| \le 4^n$. Consequently

$|R_n(x)| \le \dfrac{4^{n+1}|x-c|^{n+1}}{(n+1)!} = \dfrac{|4(x-c)|^{n+1}}{(n+1)!}$ and by an old result (Section 9.1), the limit of the term on the right-hand side is 0. So the same is true for the left-hand side, so that $\lim_{n\to\infty} R_n(x) = 0$ as wanted.

(b) Start with $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ again. In this case $f(x) = e^{-2x}$, so that $f^{(n)}(x) = (-2)^n e^{-2x}$. So, $f^{(n+1)}(z) = (-2)^{n+1} e^{-2z}$, with z between c and x. There are two possibilities: either $c < x$ (in which case $c < z < x$) or $c > x$ (in which case $x < z < c$). In the former case $e^{-2z} < e^{-2c}$, while in the latter case $e^{-2z} < e^{-2x}$. So,

$|R_n(x)| = \left| \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \right| = \dfrac{2^{n+1}e^{-2z}}{(n+1)!}|x-c|^{n+1} < \dfrac{2^{n+1}e^{-2y}}{(n+1)!}|x-c|^{n+1}$, where y is either c or x depending on the two cases just listed above. In both of these two cases e^{-2y} is a fixed number independent of n. So, $0 \le \lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty} \dfrac{2^{n+1}e^{-2y}}{(n+1)!}|x-c|^{n+1} = e^{-2y} \lim_{n\to\infty} \dfrac{|2(x-c)|^{n+1}}{(n+1)!}$ and, using the same old theorem we have used in the part (a) above, we conclude that $0 \le \lim_{n\to\infty}|R_n(x)| \le 0$, so that $\lim_{n\to\infty}|R_n(x)| = 0$, so that $\lim_{n\to\infty} R_n(x) = 0$ as wanted.