136.271

Assignment 2 (Sections 9.3, 9.4 and 9.5)
SOLUTIONS

1. [9 marks]

(a) Use the Integral Test to test if the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges. Do not forget to check that the Integral Test is applicable before you apply it.

(b) Use the Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ converges.

(c) Use the Limit Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{n^4+\sqrt{n}}$ converges.

Solution.

(a) Consider the function $f(x) = xe^{-x^2}$ for $x \geq 1$. Note first that $f(n)$ gives the general term of the series. The function is obviously positive and continuous. We now show it is decreasing by showing that the first derivative is less than 0 (for $x \geq 1$):
$f'(x) = e^{-x^2} + x(-2x)e^{-x^2} = e^{-x^2}(1-2x^2)$ and this is indeed less than 0 since e^{-x^2} is always positive, while $1-2x^2$ is negative when $x \geq 1$. So, the integral test can be used.
$\int_1^{\infty} xe^{-x^2}dx = \lim_{a\to\infty}\int_1^a xe^{-x^2}dx = \lim_{a\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_1^a = \lim_{a\to\infty}\left(-\frac{1}{2}e^{-a^2}+\frac{1}{2}e\right) = \frac{1}{2}e$, where we have used the substitution $u = x^2$ to evaluate the integral (the second equality). Since the integral converges, so does the series.

(b) First we notice that $\frac{1}{1+\sqrt{n}} \geq \frac{1}{2\sqrt{n}}$ (this is true since $1+\sqrt{n} \leq 2\sqrt{n}$ for $n \geq 1$). Since we know (theorem) that $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges, it follows that $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}}$ also diverges. By the comparison test, we have that $\sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}$ diverges too.

(c) Compare with the convergent series $\sum_{n=1}^{\infty}\frac{1}{n^2}$:

$\lim_{n\to\infty}\frac{\frac{n^2+\sqrt{n}}{n^4+\sqrt{n}}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2(n^2+\sqrt{n})}{n^4+\sqrt{n}} = \lim_{n\to\infty}\frac{n^4(1+1/n^{\frac{3}{2}})}{n^4(1+1/n^{\frac{7}{2}})} = 1$, and so the series $\sum_{n=1}^{\infty}\frac{n^2+\sqrt{n}}{n^4+\sqrt{n}}$ also converges.

2. [9 marks] Check if the following series is absolutely convergent, conditionally convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)(-5)^n}{n3^{2n}}$

(b) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{n+1}{n^2-n}\right)^n$ (the handout contained a printing error in $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{n^2-n}\right)^n$)

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n^2+1}}$

Solution.

(a) We use the Ratio test to check for absolute convergence.

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\dfrac{(n+2)5^{n+1}}{(n+1)3^{2(n+1)}}}{\dfrac{(n+1)5^n}{n3^{2n}}}=\lim_{n\to\infty}\frac{5(n+2)n}{(n+1)^2 3^2}=\frac{5}{9}\lim_{n\to\infty}\frac{(n+2)n}{(n+1)^2}=\frac{5}{9}$$ and since this is less than 1 we conclude that the series converges absolutely.

(b) Use the Root test: $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n+1}{n^2-n}\right)^n}=\lim_{n\to\infty}\frac{n+1}{n^2-n}=0$ and since this is less than 1 the series converges absolutely.

(c) We check for absolute convergence by applying the limit comparison test on $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2+1}}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2}}=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{2}{3}}}$, knowing that the latter series diverges.

$$\lim_{n\to\infty}\frac{\dfrac{1}{\sqrt[3]{n^2+1}}}{\dfrac{1}{\sqrt[3]{n^2}}}=\lim_{n\to\infty}\frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2+1}}=\lim_{n\to\infty}\sqrt[3]{\frac{n^2}{n^2+1}}=1$$, and so, both series converge or diverge simultaneously. Consequently $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2+1}}$ diverges and so the series does not converge absolutely.

To check for (conditional) convergence we use the Alternating series test.

(i) $a_{n+1}=\dfrac{1}{\sqrt[3]{n^2+2}}$ is obviously less than $a_n=\dfrac{1}{\sqrt[3]{n^2+1}}$

(ii) $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2+1}}=0$.

Consequently, by the Alternating Series Test, the series converges. So, it converges conditionally.

3. [7 marks] Find the center of convergence, the radius of convergence, and the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty}nx^n$

(b) $\sum_{n=1}^{\infty} \frac{n}{4^n}(2x-1)^n$ (the handout contained a printing error in $\sum_{n=1}^{\infty} \frac{n}{4^n(2x-1)^n}$.)

Solution.

(a) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|(n+1)x^{n+1}|}{|nx^n|} = \lim_{n\to\infty} \frac{n+1}{n}|x| = |x| \lim_{n\to\infty} \frac{n+1}{n} = |x|$. Consequently the series converges absolutely if $|x| < 1$ and it diverges if $|x| > 1$.

It remains to be seen what happens when $|x| = 1$, i.e., when $x = 1$ or $x = -1$.

Case 1. $x = 1$. In this case the series becomes $\sum_{n=1}^{\infty} n$ and this obviously diverges (Divergence Test).

Case 2. $x = -1$. In this case the series becomes $\sum_{n=1}^{\infty} n(-1)^n$ and this also diverges (Divergence Test).

So, the interval of convergence is (-1,1), the radius of convergence is 0 and the center of convergence is 0.

(b) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{n+1}{4^{n+1}}(2x-1)^{n+1}\right|}{\left|\frac{n}{4^n}(2x-1)^n\right|} = \lim_{n\to\infty} \frac{n+1}{4n}|2x-1| = \frac{|2x-1|}{4} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|2x-1|}{4}$. So, the series converges when $\frac{|2x-1|}{4} < 1$ and it diverges when $\frac{|2x-1|}{4} > 1$. Now we solve $\frac{|2x-1|}{4} < 1$ to describe it more explicitly.

$\frac{|2x-1|}{4} < 1 \Leftrightarrow |2x-1| < 4 \Leftrightarrow -4 < 2x-1 < 4 \Leftrightarrow -3 < 2x < 5 \Leftrightarrow -3/2 < x < 5/2$, where the symbol $\Leftrightarrow$ reads "is equivalent to" and stands for "means the same as".

Now we take a look what happens when $\frac{|2x-1|}{4} = 1$, i.e., when $x = -3/2$ or $x = 5/2$.

Case 1. $x = -3/2$. The series becomes $\sum_{n=1}^{\infty} \frac{n}{4^n}(2\left(-\frac{3}{2}\right)-1)^n$. We simplify a bit:

$\sum_{n=1}^{\infty} \frac{n}{4^n}(2\left(-\frac{3}{2}\right)-1)^n = \sum_{n=1}^{\infty} \frac{n}{4^n}(-4)^n = \sum_{n=1}^{\infty} n(-1)^n$ and this diverges by the divergence test.

Case 2. $x = 5/2$. The series becomes $\sum_{n=1}^{\infty} \frac{n}{4^n}(2\left(\frac{5}{2}\right)-1)^n$. We simplify a bit:

$\sum_{n=1}^{\infty} \frac{n}{4^n}(2\left(\frac{5}{2}\right)-1)^n = \sum_{n=1}^{\infty} \frac{n}{4^n}(4)^n = \sum_{n=1}^{\infty} n$ and this also diverges by the divergence test.

Conclusion: the interval of convergence is (-3/2, 5/2), the radius of convergence is 2 and the center of convergence is 1/2.