136.271 Midterm Exam

2004, February 27, 4:30-5:30

Solutions

1. [2] **(a)** State the definition of the limit of a sequence. That is, what does it mean to say that the limit of a sequence $\{a_n\}$ is the number L?

[6] **(b)** Use **only the definition** of the limit of a sequence to show that $\lim_{n\to\infty}\frac{1-n}{n-2}=-1$.

Solution. Suppose $\varepsilon>0$. We are searching for N>0, such that if $n>N$, than $\left|\frac{1-n}{n-2}-(-1)\right|<\varepsilon$. We first take a look at the last inequality and simplify it:

$$\left|\frac{1-n}{n-2}-(-1)\right|<\varepsilon \Leftrightarrow \left|\frac{1-n}{n-2}+1\right|<\varepsilon \Leftrightarrow \left|\frac{1-n+n-2}{n-2}\right|<\varepsilon \Leftrightarrow$$

$$\Leftrightarrow \left|\frac{-1}{n-2}\right|<\varepsilon \Leftrightarrow \frac{1}{n-2}<\varepsilon \Leftrightarrow \frac{1}{\varepsilon}<n-2 \Leftrightarrow \frac{1}{\varepsilon}+2<n$$

Now choose any N such that $\frac{1}{\varepsilon}+2<N$. It is now visible that if $n>N$, than $\frac{1}{\varepsilon}+2<n$ and so the inequality that we have analyzed above holds.

2. The following series do converge. Find their sums.

[6] **(a)** $\sum_{n=1}^{\infty}\frac{3^n}{5^{n-1}}$

$$\sum_{n=1}^{\infty}\frac{3^n}{5^{n-1}}=3\sum_{n=1}^{\infty}\frac{3^{n-1}}{5^{n-1}}=3\sum_{n=0}^{\infty}\frac{3^n}{5^n}=3\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n=3\frac{1}{1-\frac{3}{5}}$$

[6] **(b)** $\sum_{n=3}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Consider the partial sums associated to the $\sum_{n=3}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$:

$s_n=\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+...+\left(\frac{1}{n-2}-\frac{1}{n-1}\right)+\left(\frac{1}{n-1}-\frac{1}{n}\right)$. After (a lot of) cancellation, we see that $s_n=\frac{1}{3}-\frac{1}{n}$ and so $\lim_{n\to\infty}s_n=\frac{1}{3}$. Consequently, the sum of the series is also $\frac{1}{3}$.

3. In the following questions you are required to use specific tests. No marks will be given if you use other tests.

[5] **(a)** Use the **limit comparison test** to check if the series $\sum_{n=1}^{\infty}\frac{n+1}{n(1+n^{2/3})}$ converges.

Compare with $\sum_{n=1}^{\infty}\frac{n}{n(n^{2/3})}=\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$: $\lim_{n\to\infty}\frac{\frac{n+1}{n(1+n^{2/3})}}{\frac{1}{n^{2/3}}}=\lim_{n\to\infty}\frac{(n+1)n^{2/3}}{n(1+n^{2/3})}=1$. Since $\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$ diverges and by the limit comparison test, the original series $\sum_{n=1}^{\infty}\frac{n+1}{n(1+n^{2/3})}$ also diverges.

[5] **(b)** Use the **alternating series test** to check if the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(n+1)}$ converges.

It is obvious that $\lim_{n\to\infty}\frac{1}{\ln(n+1)}=0$ and that $\frac{1}{\ln(n+2)}<\frac{1}{\ln(n+1)}$. So the alternating test is applicable and it tells us that the series converges.

[5] **(c)** First check that **the integral test** can be used, and then use it to check if the series $\sum_{n=1}^{\infty}\frac{1}{e^n}$ converges.

We take a look at the function $f(x)=\frac{1}{e^x}$ for $x\geq 1$: it is obviously positive, continuous and decreasing. So, the integral test can be applied.

$$\int_1^{\infty}e^{-x}dx=\lim_{a\to\infty}\int_1^{a}e^{-x}dx=\lim_{a\to\infty}(-e^{-x})\Big|_1^a=\lim_{a\to\infty}(-e^{-a}+e^{-1})=\lim_{a\to\infty}(-\frac{1}{e^a}+e^{-1})=e^{-1}.$$

So the improper integral converges, and thereby the series converges too.

4. [7] Check if the following series converges absolutely, converges conditionally or diverges: $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^3+1}}$.

First we check if the series converges absolutely, that is if the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}}$ converges. Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3}}$ converges (by a theorem) and since $\frac{1}{\sqrt{n^3+1}}<\frac{1}{\sqrt{n^3}}$ it follows by the comparison test that the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}}$ converges. So, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^3+1}}$ converges absolutely.

5. (bonus [5]) Let a_n and b_n be positive numbers for all *n*. Prove that if $\sum_{n=1}^{\infty}a_n$ converges and if $\lim_{n\to\infty}b_n=0$ than $\sum_{n=1}^{\infty}a_nb_n$ also converges.

The assumption $\lim_{n\to\infty}b_n=0$ implies that for some point on, say for n>M, the members of the sequence are also less than 1. So, $a_nb_n<a_n(1)=a_n$ when n>M. So, by the comparison test (which is applicable since everything here is positive), we have that since $\sum_{n=1}^{\infty}a_n$ converges, so does $\sum_{n=1}^{\infty}a_nb_n$.