136.271 Assignment 4

Solutions

1. Find the Maclaurin series representation for the following functions and identify the interval of convergence of the series.

(a) $\dfrac{e^{2x^2}-1}{x^2}$

(b) $\sin x\cos x$ (Hint: start with $\sin 2x$)

(c) $\tan^{-1}(3x)$

(a) We know that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all numbers *x*. So $e^{2x^2} = \sum_{n=0}^{\infty}\frac{(2x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n x^{2n}}{n!}$ and thus $\frac{e^{2x^2}-1}{x^2} = \frac{1}{x^2}\left(\sum_{n=0}^{\infty}\frac{2^n x^{2n}}{n!}-1\right) = \frac{1}{x^2}\left(\sum_{n=1}^{\infty}\frac{2^n x^{2n}}{n!}\right)$ where in the last equality we have cancelled the term we get for n=0 with the –1). Further, $\frac{1}{x^2}\sum_{n=1}^{\infty}\frac{2^n x^{2n}}{n!} = \sum_{n=1}^{\infty}\frac{2^n x^{2n-2}}{n!}$ (after multiplying each term by $\frac{1}{x^2}$). So $\frac{e^{2x^2}-1}{x^2} = \sum_{n=1}^{\infty}\frac{2^n x^{2n-2}}{n!}$ and the representation is true for every number *x* except *x*=0.

(b) Recall that $\sin 2x = 2\sin x\cos x$ so that our starting function $\sin x\cos x$ is the same as $\frac{1}{2}\sin 2x$. We know that $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for every number *x*, so that $\frac{1}{2}\sin 2x = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} x^{2n+1}}{(2n+1)!}$.The representation is true for every number x.

(c) We have that $(\tan^{-1}x)' = \frac{1}{1+x^2}$. On the other hand, since $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$ for *x* in (-1,1), it follows that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ over the same interval (-1,1).

Consequently, $\tan^{-1}x = \int\frac{1}{1+x^2}dx + c = \int\sum_{n=0}^{\infty}(-1)^n x^{2n} + c$. Within the interval of convergence we can integrate term by term, so that

$\tan^{-1} x = \int\left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right)dx + c = \sum_{n=0}^{\infty}\int(-1)^n x^{2n}dx + c = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + c$. Since $\tan^{-1} 0 = 0$ and since the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ is also 0 when x=0, it follows that the constant c is 0. So, $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ for x in the interval (-1,1), and so $\tan^{-1} 3x = \sum_{n=0}^{\infty}(-1)^n \frac{(3x)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{3^{2n+1}x^{2n+1}}{2n+1}$ for $3x$ in (-1,1), i.e., for x in the interval $(-\frac{1}{3},\frac{1}{3})$.

2. Find the Taylor series representation of the function $\ln x$ centered at a=3.

Taylor's formula tells us that $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$. In this question $f(x) = \ln x$ and a=3. We compute: $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2\cdot 1}{x^3}$, $f^{(4)}(x) = -\frac{3\cdot 2\cdot 1}{x^4}$, and, in general, $f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$. So, $f^{(n)}(3) = (-1)^{n+1}\frac{(n-1)!}{3^n}$ (with $f^{(0)}(3) = f(3) = \ln 3$) and, after substituting this in the Taylor's formula, we get

$f(x) = \ln 3 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\frac{(n-1)!}{3^n}}{n!}(x-3)^n$. We simplify this a bit to get our final answer:

$f(x) = \ln 3 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3^n n}(x-3)^n$.

3. Use multiplication of series to find the first three nonzero terms of the Maclaurin series representation of the function $\ln(2+x)\cdot\tan^{-1}\left(x^2\right)$.

First, $\ln(1+x) = \int\frac{1}{1+x}dx + c = \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx + c$, where we have used the fact that $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$ for $-1<x<1$. Continuing,

$\int\left(\sum_{n=1}^{\infty}(-1)^n x^n\right)dx + c = \sum_{n=0}^{\infty}\int(-1)^n x^n dx \ + c = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \ + c$. So that

$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \ + c$. Substituting x=0, we get that c=0. So

$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$. Now, $\ln(2+x) = \ln[2(1+x/2)] = \ln 2 + \ln(1+x/2)$ and since

$\ln(1+x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$, we have that

$\ln(1+x/2)=\sum_{n=0}^{\infty}(-1)^n\frac{(x/2)^{n+1}}{n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{2^{n+1}(n+1)}$. Summarizing,

$\ln(2+x)=\ln 2+\ln(1+x/2)=\ln 2+\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{2^{n+1}(n+1)}$.

Now we pay attention to $\tan^{-1}\left(x^2\right)$. Since $\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$, we get that

$\tan^{-1}\left(x^2\right)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(x^2\right)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$.

Finally we take a look at the product: $\ln(2+x)\cdot\tan^{-1}\left(x^2\right)$:

$$\ln(2+x)\cdot\tan^{-1}\left(x^2\right)=\left[\ln 2+\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{2^{n+1}(n+1)}\right]\cdot\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}\right]=$$

$$=\left[\ln 2+\frac{x}{2}-\frac{x^2}{2^3}+\dots\right]\left[x^2-\frac{x^6}{3}+\dots\right]=(\ln 2)x^2-\frac{x^3}{2}-\frac{x^4}{2^3}+\dots$$

4. Use power series to evaluate $\int_0^x\cos(t^2)dt$.

Since $\cos t=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}$, we have that $\cos t^2=\sum_{n=0}^{\infty}\frac{(-1)^n\left(t^2\right)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n t^{4n}}{(2n)!}$. So,

$$\int_0^x\cos(t^2)dt=\int_0^x\sum_{n=0}^{\infty}\frac{(-1)^n t^{4n}}{(2n)!}dt=\sum_{n=0}^{\infty}\int_0^x\frac{(-1)^n t^{4n}}{(2n)!}dt=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}.$$

5.

a) Use binomial series to find the power series representation of the function $\frac{1}{\sqrt{4+x^2}}$. Simplify your answer.

(**b**) Use your answer in (a) to compute the sum of $\sum_{n=1}^{\infty}(-1)^n\frac{(1)(3)\cdots(2n-1)}{2^{5n+1}n!}$.

(a) We have: $\dfrac{1}{\sqrt{4+x^2}} = \dfrac{1}{2\sqrt{1+\left(x/2\right)^2}} = \dfrac{1}{2}\left(1+\left(x/2\right)^2\right)^{-1/2}$. Now we take a look at the associated binomial series $(1+x)^{-1/2} = 1+\sum_{n=1}^{\infty}\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}x^n$.

Consequently, $\left(1+\left(x/2\right)^2\right)^{-1/2} = 1+\sum_{n=1}^{\infty}\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}\left(\left(\frac{x}{2}\right)^2\right)^n$. So

$\dfrac{1}{2}\left(1+\left(x/2\right)^2\right)^{-1/2} = \dfrac{1}{2}+\dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}\left(\left(\frac{x}{2}\right)^2\right)^n$ and we start simplifying:

$$\frac{1}{2}\left(1+\left(x/2\right)^2\right)^{-1/2} = \frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(1)(1+2)(1+4)\ldots(1+2n-2)}{2^n\, n!}\frac{x^{2n}}{2^{2n}} = \frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n(1)(3)(5)\ldots(2n-1)}{2^{3n+1}n!}x^{2n}$$

(b) We observe that we get $\sum_{n=1}^{\infty}(-1)^n\dfrac{(1)(3)\cdots(2n-1)}{2^{5n+1}n!}$ by substituting $x=\dfrac{1}{2}$ in $\sum_{n=1}^{\infty}\dfrac{(-1)^n(1)(3)(5)\ldots(2n-1)}{2^{3n+1}n!}x^{2n}$. But, from part (a) we find that $\dfrac{1}{2}\left(1+\left(x/2\right)^2\right)^{-1/2}-\dfrac{1}{2} = \sum_{n=1}^{\infty}\dfrac{(-1)^n(1)(3)(5)\ldots(2n-1)}{2^{3n+1}n!}x^{2n}$, and so, after substituting $x=\dfrac{1}{2}$, we get $\sum_{n=1}^{\infty}(-1)^n\dfrac{(1)(3)\cdots(2n-1)}{2^{5n+1}n!} = \dfrac{1}{2}\left(1+\left(\frac{1/2}{2}\right)^2\right)^{-1/2}-\dfrac{1}{2}$.