136.271 Assignment 3

Due March 13, 2004, (Solutions)

1. Which of the following series converges absolutely, which converges conditionally and which diverges? Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0.1)^n}{n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \sqrt[n]{10}$

(d) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \left(\frac{\ln n}{\ln(n^2)}\right)^n$

(a) $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}(0.1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{(0.1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n(10)^n}$ and since $\displaystyle\frac{1}{n(10)^n} < \frac{1}{(10)^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(10)^n}$ converges, we have by the comparison test that $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}(0.1)^n}{n}\right|$ converges. Consequently the series in this problem converges absolutely.

(b) $\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-2)^{n+1}}{n+5^n}\right| = \sum_{n=2}^{\infty} \frac{2^{n+1}}{n+5^n} = 2\sum_{n=2}^{\infty} \frac{2^n}{n+5^n}$, since $\displaystyle\frac{2^n}{n+5^n} < \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n$ and since $\displaystyle\sum_{n=2}^{\infty} \left(\frac{2}{5}\right)^n$ converges, the original series converges absolutely.

(c) $\displaystyle\lim_{n\to\infty} \sqrt[n]{10} = \lim_{n\to\infty} 10^{\frac{1}{n}} = 10^0 = 1$ and so $\displaystyle\lim_{n\to\infty} (-1)^{n+1} \sqrt[n]{10}$ is not 0. Thereby the series diverges (by the divergence test).

(d) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \left(\frac{\ln n}{\ln(n^2)}\right)^n = \sum_{n=2}^{\infty} (-1)^n \left(\frac{\ln n}{2\ln n}\right)^n = \sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n$ and since $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$ converges, the series in this problem converges absolutely.

2. Find the interval of convergence for the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

(b) $\sum_{n=0}^{\infty} \frac{(2x+3)^{2n+1}}{n!}$

(c) $\sum_{n=1}^{\infty} (n)^n x^n$

For the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ (in part **a** above), fund the sum of the series as a function on x.

(a) We use the ratio test (the root test is also easy to use here):

$\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{10^{n+1}}}{\frac{(x-2)^n}{10^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)}{10} \right| = \left| \frac{(x-2)}{10} \right|$ and so the series converges when $\left| \frac{(x-2)}{10} \right| < 1$, which, after solving yields $-8 < x < 12$. When $x = 12$ the series becomes $\sum_{n=0}^{\infty} \frac{(12-2)^n}{10^n} = \sum_{n=0}^{\infty} 1$ which obviously diverges, while when $x = -8$ the series becomes $\sum_{n=0}^{\infty} \frac{(-8-2)^n}{10^n} = \sum_{n=0}^{\infty} (-1)^n$ which also diverges. Consequently, the interval of convergence of this series is $(-8\ ,\ 12)$.

We immediately take care of the last part of this problem and find the sum of this series over the interval of convergence: $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{10} \right)^n = \frac{1}{1 - \frac{x-2}{10}}$.

(b) Use the ratio test again: $\lim_{n\to\infty} \left| \frac{\frac{(2x+3)^{2n+3}}{(n+1)!}}{\frac{(2x+3)^{2n+1}}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(2x+3)^2}{n+1} \right| = 0$ and since that is always less than 1 the series $\sum_{n=0}^{\infty} \frac{(2x+3)^{2n+1}}{n!}$ converges always and the interval of convergence if $(-\infty, +\infty)$.

(c) Use the root test: $\lim_{n\to\infty} \sqrt[n]{|(nx)^n|} = \lim_{n\to\infty} |nx| = \infty$ with the last equality being true for all x except for x=0. Since the limit is never less than 1 for x not equal to 0, the series $\sum_{n=1}^{\infty} (n)^n x^n$ converges only when x=0 and the interval of convergence is [0,0].

3. Find the power series representation of the function $g(x) = \frac{x^2}{(1+2x)^2}$ and the interval of convergence of the power series.

First we focus on the function $h(x) = \frac{1}{(1+2x)^2}$. Observe first that $h(x) = \frac{1}{2}\left(\frac{1}{1+2x}\right)'$.

Now, from what we know, $\frac{1}{1+2x} = \sum_{n=0}^{\infty}(-2x)^n$ and the series converges for $|2x| < 1$, which gives $-\frac{1}{2} < x < \frac{1}{2}$. Consequently, $h(x) = \frac{1}{2}\left(\frac{1}{1+2x}\right)' = \frac{1}{2}\left(\sum_{n=0}^{\infty}(-2x)^n\right)'$ and within $-\frac{1}{2} < x < \frac{1}{2}$ we can differentiate term by term to get $\frac{1}{2}\left(\sum_{n=0}^{\infty}(-2x)^n\right)' = \frac{1}{2}\sum_{n=1}^{\infty}(-2)^n nx^{n-1}$ (note the change of the start of the counter: the term we get for n=0 in the series we differentiate is annihilated after differentiation). So, summarizing, $h(x) = \frac{1}{2}\sum_{n=1}^{\infty}(-2)^n nx^{n-1}$ for $-\frac{1}{2} < x < \frac{1}{2}$. Our original function was $x^2h(x)$, and so $x^2h(x) = \frac{1}{2}x^2\sum_{n=1}^{\infty}(-2)^n nx^{n-1} = \sum_{n=1}^{\infty}(-2)^{n-1}nx^{n+1}$ over $-\frac{1}{2} < x < \frac{1}{2}$ (where in the second equality we have simply multiplied every term of the series by $\frac{1}{2}x^2$.