## THE UNIVERSITY OF MANITOBA

DATE: Oct. 27, 2004
DEPARTMENT \& COURSE NO: 136.270
EXAMINATION: Calculus 3A

NAME: (PRINT)
STUDENT NUMBER: $\qquad$
SIGNATURE:
(I understand that cheating is a serious offense)

## INSTRUCTIONS TO THE STUDENT

This is a one-hour exam. There are 3 pages of questions and one blank page for rough work. Check now that you have all 3 pages of questions.. Answer all the questions in the spaces provided. If necessary, you may continue your work on the reverse sides of the pages but PLEASE INDICATE CLEARLY that your work continues and where the continuation may be found. DO NOT detach any question pages from the exam.

The point value of each question is indicated to the left of the question number. The maximum score possible is 60 points.

Please present your work CLEARLY and LEGIBLY, and use a pen (not red) or a dark pencil. Justify your answers unless otherwise stated.

NO CALCULATORS, TEXTS, NOTES OR OTHER AIDS ARE PERMITTED.

## Values

1. In all of (a), (b), (c), (d) and (e) below, we consider the vector function $\mathbf{r}(t)=(2 \cos t, t \sqrt{5}, 2 \sin t)$.
[5] (a) Find the length of the arc between the points corresponding to $t=0$ and $t=\pi$.

Solution: $s=\int_{0}^{\pi}\left|\mathbf{r}^{\prime}(t)\right| d t=\int_{0}^{\pi} \sqrt{4 \sin ^{2} t+4 \cos ^{2} t+5} d t=3 t \left\lvert\, \begin{aligned} & \pi \\ & 0\end{aligned}=3 \pi\right.$
[4] (b) Sketch $\mathbf{r}(t)=(2 \cos t, t \sqrt{5}, 2 \sin t)$.

## Solution.


[5] (c) Find the unit tangent vector at the point $(2,0,0)$.

## Solution.

$\mathbf{r}^{\prime}(t)=(-2 \sin t, \sqrt{5}, 2 \cos t)$ (found in (a) too), and since the given point happens when $t=0$ we have that $\mathbf{r}^{\prime}(0)=(0, \sqrt{5}, 2)$. The length of this vector is $\left|\mathbf{r}^{\prime}(0)\right|=\sqrt{5+4}=3$ and so the unit tangent vector at the given point is $\mathbf{T}(0)=\frac{1}{3}(0, \sqrt{5}, 2)$.
[4] (d) Find one point on the curve defined by the vector function $\mathbf{r}(t)$ where the normal plane is perpendicular to the unit tangent vector found in (c) above.
Solution. We have one such point already: part (c) tells us that $(2,0,0)$ is such a point! Otherwise (if that the above approach is overlooked we solve $\mathbf{r}^{\prime}(t)=k \frac{1}{3}(0, \sqrt{5}, 2)$, i.e., $(-2 \sin t, \sqrt{5}, 2 \cos t)=k \frac{1}{3}(0, \sqrt{5}, 2)$.
[6] (e) Find the bi-normal vector at the point found in (d) above.
Solution. We need the unit normal vector $\mathbf{N}(t)=\frac{\mathbf{T}^{\prime}(t)}{\left|\mathbf{T}^{\prime}(t)\right|}$ at point when $t=0$. We have found above that $\mathbf{r}^{\prime}(t)=(-2 \sin t, \sqrt{5}, 2 \cos t)$ and that $\left|\mathbf{r}^{\prime}(t)\right|=3$. So $\mathbf{T}(t)=\frac{1}{3}(-2 \sin t, \sqrt{5}, 2 \cos t)$. We differentiate to get $\mathbf{T}^{\prime}(t)=\frac{1}{3}(-2 \cos t, 0,-2 \sin t)$. At $t=0$ we have $\mathbf{T}^{\prime}(0)=\frac{1}{3}(-2,0,0)$ and the length of this vector is obviously $\frac{2}{3}$.
Consequently $\mathbf{N}(0)=(-1,0,0)$. The bi-normal vector is the cross product of
$\mathbf{T}(0)=\frac{1}{3}(0, \sqrt{5}, 2)$ and $\mathbf{N}(0)=(-1,0,0)$ : we compute it to get $\mathbf{B}(0)=\frac{1}{3}(0,-2, \sqrt{5})$.
2. Show that the following limits do not exist.
(a) $\lim _{(x, y) \rightarrow(0,0)} \frac{x y}{x^{2}+y^{2}}$

Solution. Along $x=0$ the limit is 0 , while along $y=x$ it is $\frac{1}{2}$. That shows the limit does not exist.

$$
\begin{equation*}
\text { (b) } \lim _{(x, y) \rightarrow(0,0)} \frac{2 x y^{2}}{x^{2}+y^{4}} \tag{7}
\end{equation*}
$$

Solution. Along $x=0$ the limit is 0 . Along $x=y^{2}$ the limit is 1 and so the limit does not exist.

## Values

[3]
3. (a) State the definition of $\lim _{(x, y) \rightarrow(a, b)} f(x, y)=L$.

Solution. ... (see lectures/textbook)
[7] (b) Use the definition if limit (part (a) above) to show that $\lim _{(x, y) \rightarrow(0,0)} 4 x^{2}+4 y^{2}+\frac{1}{3}=\frac{1}{3}$. No points will be awarder if other methods are used.
Solution. We want to show that for every $\varepsilon>0$ there is some $\delta>0$ such that if $\sqrt{x^{2}+y^{2}}<\delta$ then $\left|\frac{1}{4 x^{2}+4 y^{2}}+\frac{1}{3}-\frac{1}{3}\right|<\varepsilon$. We simplify the last inequality a bit (the symbol $\Leftrightarrow$ stands for "means the same as"):
$\left|4 x^{2}+4 y^{2}+\frac{1}{3}-\frac{1}{3}\right|<\varepsilon \Leftrightarrow\left|4 x^{2}+4 y^{2}\right|<\varepsilon \Leftrightarrow 4 x^{2}+4 y^{2}<\varepsilon$. Going a bit further we get: $4 x^{2}+4 y^{2}<\varepsilon \Leftrightarrow x^{2}+y^{2}<\frac{\varepsilon}{4} \Leftrightarrow \sqrt{x^{2}+y^{2}}<\sqrt{\frac{\varepsilon}{4}}$. Now we see that if we choose $\delta=\sqrt{\frac{\varepsilon}{4}}$ then $\sqrt{x^{2}+y^{2}}<\delta$ would imply $\sqrt{x^{2}+y^{2}}<\sqrt{\frac{\varepsilon}{4}}$ and so, going back along $\Leftrightarrow$-s, we will get $\left|\frac{1}{4 x^{2}+4 y^{2}}+\frac{1}{3}-\frac{1}{3}\right|<\varepsilon$ as wanted.
[4] (c) Use the Squeeze theorem and the result of part (b) to show that $\lim _{(x, y) \rightarrow(0,0)} 3 x^{2}+4 y^{2}+\frac{1}{3}=\frac{1}{3}$. No points will be given if other methods are used.
Solution. Notice that $\frac{1}{3} \leq 3 x^{2}+4 y^{2}+\frac{1}{3} \leq 4 x^{2}+4 y^{2}+\frac{1}{3}$, By the Squeeze theorem we have $\lim _{(x, y) \rightarrow(0,0)} \frac{1}{3} \leq \lim _{(x, y) \rightarrow(0,0)} 3 x^{2}+4 y^{2}+\frac{1}{3} \leq \lim _{(x, y) \rightarrow(0,0)} 4 x^{2}+4 y^{2}+\frac{1}{3}$. Since the two limit at the ends of the inequalities are both $\frac{1}{3}$ (one of them done in (b) above), it follow that $\lim _{(x, y) \rightarrow(0,0)} 3 x^{2}+4 y^{2}+\frac{1}{3}=\frac{1}{3}$ too.
4. Given that $f(x, y, z)=x(\sin z) e^{x y}$
[5] (a) Find all three first partial derivatives.
Sol. $\frac{\partial f}{\partial x}(x, y, z)=\sin z\left[e^{x y}+x y e^{x y}\right], \frac{\partial f}{\partial y}(x, y, z)=x^{2}(\sin z) e^{x y}$ and $\frac{\partial f}{\partial z}(x, y, z)=x(\cos z) e^{x y}$
[4]
(b) Find $\frac{\partial^{2} f}{\partial z \partial y}(1,1, \pi)$ (or was it $\frac{\partial^{2} f}{\partial y \partial x}(1,1, \pi)$ ?).
$\frac{\partial^{2} f}{\partial z \partial y}(x, y, z)=x^{2}(\cos z) e^{x y}$ and so $\frac{\partial^{2} f}{\partial z \partial y}(1,1, \pi)=-e$.

