136.270

Assignment 3 (Sections 14.4, 14.5, 14.6)

Solutions

1. [5 marks] [2]**(a)** Find $\frac{\partial w}{\partial v}$ when $u=1$ and $v=2$ if $w=xy+\ln z$, $x=\frac{v^2}{u}$, $y=u+v$ and $z=\cos u$.

[3] **(b)** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point (1,1,1) if z is the function on x and y defined by the equation $z^3-xy+yz=2-y^3$.

Solution. **(a)** $\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}=y\frac{2v}{u}+x+\frac{1}{z}0=(u+v)\frac{2v}{u}+\frac{v^2}{u}$. So, when $u=1$ and $v=2$ we find $\frac{\partial w}{\partial v}=(1+2)\frac{4}{1}+\frac{4}{1}=16$.

(b) Denote $F(x,y,z)=z^3-xy+yz+y^3-2$. Then z is defined by the equation $F(x,y,z)=0$. It follows from our theory that $\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$, We compute:

$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{-y}{3z^2+y}$ and at the point (1,1,1) we have $\frac{\partial z}{\partial x}=\frac{1}{4}$. Similarly

$\frac{\partial z}{\partial y}=-\frac{-x+z+3y^2}{3z^2+y}$ and so $\frac{\partial z}{\partial y}=-\frac{3}{4}$ at the given point.

2. [7 marks] [4] **(a)** Find the point on the surface $z=x^3y+y$ where the tangent plane is parallel to the plane $-3x-2y+z=1$. Then find the equation of the tangent plane at that point.

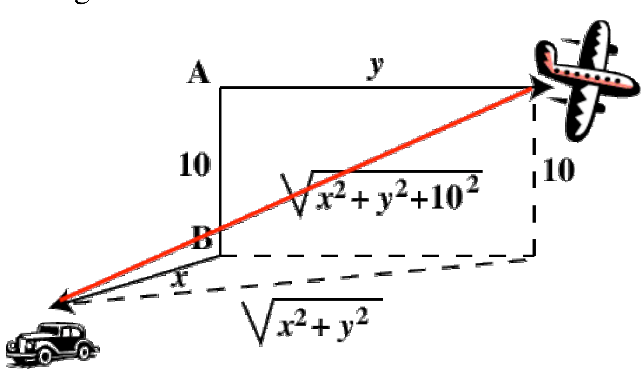
[3] **(b)** Find the equation of the normal line to the surface $z=x^3y+y$ at the point (1,2,4).

Solution. **(a)** The tangent plane of $z=x^3y+y$ has $(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},-1)$ as a normal vector, while the given plane is normal to the vector $(-3,-2,1)$. The two planes are parallel if the two normal vectors are parallel, and the latter happens if one of them is a multiple of the other. So, we are searchin for the point (x,y,z) where $(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},-1)=k(-3,-2,1)$. The third coordinates tell us that $k=-1$, while the first two (together with $k=-1$) yield the system $3x^2y=3$, $x^3+1=2$. The second equation gives $x=1$ and with that we find that $y=1$ too. So, the point we wanted is $(1,1,2)$ where the thid coordinate is found by substituting $x=1$ and $y=1$ in $z=x^3y+y$.

(b) Since the surface is defined by $-z+x^3y+y=0$, the normal line to that surface is parallel to the gradient vector ∇F where $F(x,y,z)=-z+x^3y+y$. We compute $\nabla F=(3x^2y,x^3+1,-1)$ and at the given point we have $\nabla F=(6,2,-1)$. So, the equation of that line is $x=1+6t$, $y=2+2t$ and $z=4-t$.

3. [6 marks] At 10:00 a.m. a plane traveling east is 10 kilometers above a southbound car. The plane travels horizontally at 500 km/hour while the car maintains a constant speed of 100 km/hour. How fast is the distance between the plane and the car increasing at 11:00 a.m.? Do NOT simplify your answer.

Solution. At 10:00 the car is at the point A while the plane is at the point B 10 km straight above A. Denote the distance between the plane and the car by z, denote that distance between the plane and the point B by y and denote the distance between the car and the point A by x. The picture tells is that $z^2=x^2+y^2+10^2$. So, $z=\sqrt{x^2+y^2+10^2}$. We now differentiate z with respect to the time t.



$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\frac{x}{\sqrt{x^2+y^2+100}}\frac{dx}{dt}+\frac{y}{\sqrt{x^2+y^2+100}}\frac{dy}{dt}$$. At 11:00 we have that y=500, x=100, so that, after substituting, we get

$$\frac{\partial z}{\partial t}=\frac{100}{\sqrt{100^2+500^2+100}}100+\frac{500}{\sqrt{100^2+500^2+100}}500 \text{ km/hour}$$

4. [7 marks][3.5] **(a)** Find the directional derivative of the function $f(x,y,z)=xy+yz+zx$ at the point $(1,-1,2)$ in the direction of the vector $(3,6,-2)$.

[3.5] **(b)** Find the direction in which the function $f(x,y,z)=xy+yz+zx$ increases the most rapidly at the point $(1,-1,2)$. In which direction the same function decreases the most rapidly at $(1,-1,2)$? Find the maximal rate of change of $f(x,y,z)=xy+yz+zx$ at the point $(1,-1,2)$.

Solution. **(a)** First we need the unit vector in the direction of $(3,6,-2)$:

$\mathbf{u}=\dfrac{1}{\sqrt{3^2+6^2+2^2}}(3,6,-2)=\dfrac{1}{7}(3,6,-2)$. By definition, we have:

$\mathbf{D_u}f(1,-1,2)=\nabla f(1,-1,2)\bullet(\frac{3}{7},\frac{6}{7},-\frac{2}{7})=(\frac{\partial f}{\partial x}(1,-1,2),\frac{\partial f}{\partial y}(1,-1,2),\frac{\partial f}{\partial z}(1,-1,2))\bullet(\frac{3}{7},\frac{6}{7},-\frac{2}{7})$.

We compute $\frac{\partial f}{\partial x}(x,y,z)=y+z$, $\frac{\partial f}{\partial x}(1,-1,2)=1$, $\frac{\partial f}{\partial y}(x,y,z)=x+z$, $\frac{\partial f}{\partial y}(1,-1,2)=3$, $\frac{\partial f}{\partial z}(x,y,z)=y+x$, $\frac{\partial f}{\partial z}(1,-1,2)=0$. Hence $\mathbf{D_u}f(1,-1,2)=(1,3,0)\bullet(\frac{3}{7},\frac{6}{7},-\frac{2}{7})=3$.

(b) The function $f(x,y,z)=xy+yz+zx$ increases the most rapidly in the direction of the gradient vector at the given point. We have found in (a) above that $\nabla f(1,-1,2)=(1,3,0)$. The function decreases the most rapidly in the opposite direction of the gradient vector, that is, in the direction of $-\nabla f(1,-1,2)=(-1,-3,0)$. The maximal rate of change is the length of the gradient vector $\|\nabla f(1,-1,2)\|=\|(1,3,0)\|=\sqrt{10}$.