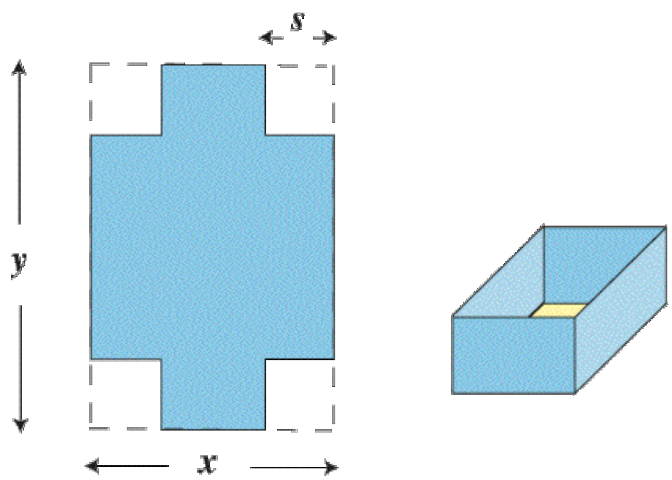
136.270 Solutions

Assignment 4 (Sections 14.7, 14.8, 15.1, 15.2, 15.3)

Handed: November 24, 2004. Due: December 1, 2004 in class.
Show your work. Providing answers without justifying them will not be sufficient.

1. An open-top rectangular box of a specified volume V is to be constructed from a sheet of metal by first cutting a rectangular piece of the sheet, then cutting equal squares from the corners, folding up the remaining flaps and soldering their edges together (see the picture). Find the dimensions of the material (*x*, *y* and *s* in the picture) that minimize



the amount of the sheet metal used.

Solution. We are minimizing $A=(x-2s)(y-2s)+2(x-2s)s+2(y-2s)s$ under the condition $V=(x-2s)(y-2s)s$. Set $u=x-2s$, $v=y-2s$. Then $V=uvs$ and $A=uv+2us+2vs$. Solving $V=uvs$ for *s* gives $\frac{V}{uv}=s$. The area then becomes a function on *u* and *v*: $A=uv+(2u+2v)s=uv+(2u+2v)\frac{V}{uv}=uv+2V(\frac{1}{u}+\frac{1}{v})$. Compute the partial derivatives: $\frac{\partial A}{\partial u}=v-2V\frac{1}{u^2}$ and $\frac{\partial A}{\partial u}=u-2V\frac{1}{v^2}$. Equate these to 0 to find the critical points; get the system $0=v-2V\frac{1}{u^2}$ and $0=u-2V\frac{1}{v^2}$. The solution is $u=\sqrt[3]{2V}$ and $v=\sqrt[3]{2V}$. The second derivative test yields a local minimum at that point. Since it is the only local minimum and since the function A is a polynomial it follows that that point gives the absolute minimum. We can then compute from above that $s=\frac{1}{2}\sqrt[3]{2V}$, $x=2\sqrt[3]{2V}$ and $y=2\sqrt[3]{2V}$ - the dimensions of the smallest (in area) such box.

2. The plane $x+y+z=1$ is heated and the temperature at any point is given by $T(x,y,z)=4-2x^2-y^2-z^2$. Use Lagrange multipliers to find the hottest point on the plane.

Solution. We are maximizing the function $T(x,y,z)=4-2x^2-y^2-z^2$ under the constraint $x+y+z=1$. The critical point come from solving $\nabla T=\lambda\nabla g$ and $x+y+z=1$, where $g(x,y,z)=x+y+z$. Computing the two gradients and equating component-wise gives $-4x=\lambda$, $-2y=\lambda$, $-2z=\lambda$, i.e., $x=-\dfrac{\lambda}{4}$, $y=-\dfrac{\lambda}{2}$ and $z=-\dfrac{\lambda}{2}$. Substitute all these in $x+y+z=1$ to get $-\dfrac{5}{4}\lambda=1$, i.e., $\lambda=-\dfrac{4}{5}$. So, $x=\dfrac{1}{5}$, $y=\dfrac{2}{5}$, $z=\dfrac{2}{5}$. This is the only critical point. So, it has to yield an absolute extremum. It follows from the meaning of the quantities we discuss (i.e., from the shape of $T(x,y,z)=4-2x^2-y^2-z^2$) that must be the absolute maximum.

3. **(a)** Evaluate $\iint\limits_{\mathbf{R}} x\, dA$ where **R** is the region in the first quadrant bounded by $y=0,\ y=2,\ x=0$, and $x=1+y^2$.

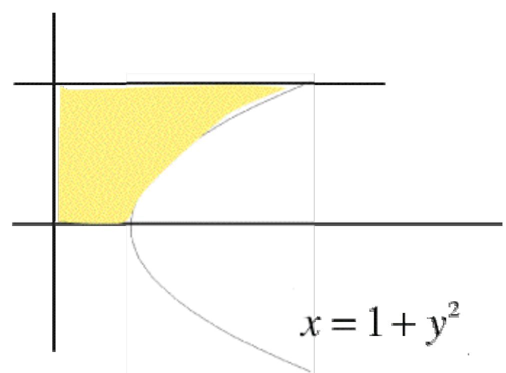
(b) Evaluate $\int_0^1\int_x^1 e^{y^2}\,dy\,dx$.

(c) Find the volume of the solid bounded by the paraboloid $z=x^2+y^2$ and the plane $z=1$.

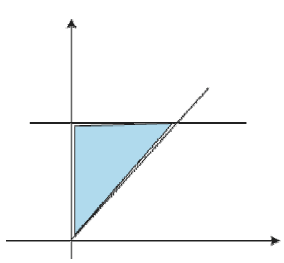
Solution. **(a) R** can be described by the following two pairs of inequalities:

$$0\le y\le 2$$
$$0\le x\le 1+y^2$$

So,



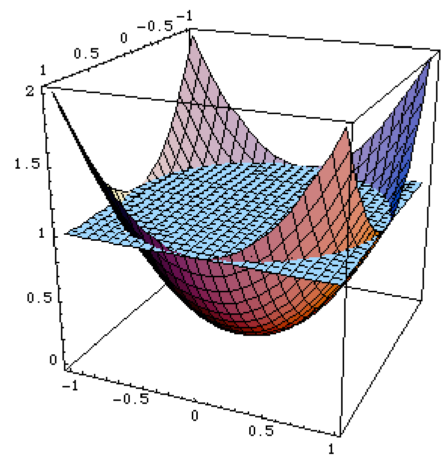
$$\iint\limits_{\mathbf{R}} x\,dA=\int_0^2\int_0^{1+y^2} x\,dx\,dy=\int_0^2\left(\frac{x^2}{2}\Bigg|_0^{1+y^2}\right)dy=\int_0^2\frac{(1+y^2)^2}{2}\,dy=\frac{1}{2}\left(y+2\frac{y^3}{3}+\frac{y^5}{5}\Bigg|_0^2\right)=\frac{1}{2}\left(2+\frac{2^3}{3}+\frac{2^5}{5}\right)$$



(b) Here is the region of integration (picture). The inner integral of iterated integral in the statement of the problem is not expressible in terms of basic functions. So, we need to change the order of the integration, i.e., we need to express the region of integration in such a way that the inner integration is with respect to x (rather than y). Here it is: $0 \le y \le 1$, $0 \le x \le y$. Then the iterated integral becomes $\int_0^1 \int_0^y e^{y^2} dx\ dy$. We can now solve it:

$$\int_0^1 \int_0^y e^{y^2} dx\ dy = \int_0^1 y e^{y^2} dy = \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{1}{2}(e-1).$$

(c) The solid S is shown in the picture. It is bounded by the surface $z = x^2 + y^2$ from blow and by the plane $z = 1$ from above. We can find its volume by computing the volume of the solid S_1 outside S and inside the cylinder $1 = x^2 + y^2$, and then subtracting the latter from the volume of the cylinder of height 1 and over the base $1 = x^2 + y^2$ in the xy-plane. The volume of the last cylinder is π (its radius is 1 and its height is also 1). The volume of S_1 is $\iint_{\mathbf{R}} (x^2 + y^2)\ dA$ where **R** is the disk $1 \ge x^2 + y^2$. Because of symmetry we can focus on the first octant and the associated quarter of a disk, which can be described by the inequalities $0 \le x \le 1$



and $0 \le y \le \sqrt{1-x^2}$. So, its volume is $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\ dy\ dx$. We now compute:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\ dy\ dx = \int_0^1 (x^2 y + \frac{y^3}{3}) \Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 (x^2\sqrt{1-x^2} + \frac{(1-x^2)\sqrt{1-x^2}}{3})\ dx =$$

$$= \frac{1}{3}\int_0^1 (2x^2\sqrt{1-x^2} + \sqrt{1-x^2})\ dx = \frac{1}{3}\int_0^1 2x^2\sqrt{1-x^2} dx + \frac{1}{3}\int_0^1 \sqrt{1-x^2} dx$$

The two integrals are doable with one trigonometric substitution $x = \sin t$:

$\int x^2\sqrt{1-x^2} dx = \frac{1}{8}(x\sqrt{1-x^2}(2x^2-1) + \sin^{-1} x)$ and $\frac{1}{3}\int_0^1 2x^2\sqrt{1-x^2} dx = \frac{\pi}{24}$;

$\int \sqrt{1-x^2} dx = \frac{1}{2}(x\sqrt{1-x^2} + \sin^{-1} x)$ and $\frac{1}{3}\int_0^1 \sqrt{1-x^2} dx = \frac{\pi}{12}$. So, the total volume of one quarter of S_1 is $\frac{3\pi}{24}$. Consequently the volume of S_1 is $4\frac{3\pi}{24} = \frac{3\pi}{6} = \frac{\pi}{2}$, and so the volume of S is $\pi - \frac{\pi}{2} = \frac{\pi}{2}$.