136.270

Assignment 2 Solutions (Sections 13.3, 14.1, 14.2)

Handed: October 15, 2004. Due: October 22, 2004 in class

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1. [6 marks] [3]**(a)** Find the length of the curve $\vec{\mathbf{r}}(t) = (1-2t, 4t-1, t-2)$ between the points $(1,-1,-2)$ and $(-3,7,0)$.

[3] **(b)** Reparametrize the curve in part (a) with respect to arc length measured from the point where $t = 0$ (in the direction of increasing *t*).

Solution: (a) Observe first that $(1,-1,-2)$ happens when $t = 0$ and that $(-3,7,0)$ happens when $t = 2$. So, the desired arc length is

$$s = \int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 |(1-2t, 4t-1, t-2)'|\,dt = \int_0^2 |(-2,4,1)|\,dt = \int_0^2 \sqrt{4+16+1}\,dt = 2\sqrt{21}.$$

(b) $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{21}\,du = t\sqrt{21}$ so that $t = \dfrac{s}{\sqrt{21}}$. Substitute this for *t* in the vector function to get $\vec{\mathbf{r}}(s) = \left(1 - 2\dfrac{s}{\sqrt{21}}, 4\dfrac{s}{\sqrt{21}} - 1, \dfrac{s}{\sqrt{21}} - 2\right)$.

2. [7 marks] [4] (a) Find the equation of the osculating plane of the curve $\vec{\mathbf{r}}(t) = (1, t^2, t)$ at the point $(1,1,1)$

[3] **(b)** At what point(s) (if any) on the curve $\mathbf{r}(t) = (1, t^2, t)$ is the normal plane parallel to the plane $6y + 3z = -3$?

Solution. (a) Notice before we start with the computation that the point $(1,1,1)$ happens when $t = 1$. The osculating plane is perpendicular to the bi-normal vector, which is the cross product $\mathbf{T}(t) \times \mathbf{N}(t)$. First we find $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{(1,t^2,t)'}{|\mathbf{r}'(t)|} = \dfrac{(0,2t,1)}{\sqrt{4t^2+1}}$. Digress a bit to notice that $\mathbf{T}(1) = \dfrac{(0,2,1)}{\sqrt{5}}$ and so $(0,2,1)$ is parallel to $\mathbf{T}(1)$. The normal vector can then be found by computing $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\left(\dfrac{(0,2t,1)}{\sqrt{4t^2+1}}\right)'}{\left|\left(\dfrac{(0,2t,1)}{\sqrt{4t^2+1}}\right)'\right|}$. That could be tedious. We look for shortcuts. Back up a bit to notice that we need a vector (any vector!) that is perpendicular to the osculating plane. Since $\mathbf{T}'(t)$ is parallel to $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, the cross product of $(0,2,1)\times \mathbf{T}'(1)$ will be parallel to the bi-normal vector. So, we start with

$\mathbf{T}'(t)$: $\mathbf{T}'(t)=\left(\dfrac{(0,2t,1)}{\sqrt{4t^2+1}}\right)' = \dfrac{(0,2,0)\sqrt{4t^2+1}-(0,2t,1)\dfrac{8t}{2\sqrt{4t^2+1}}}{4t^2+1}$. We do not want to simplify this: we simply substitute t=1 (the moment when the given point happens) to get

$$\mathbf{T}'(1)=\frac{(0,2,0)\sqrt{5}-(0,2,1)\dfrac{8}{2\sqrt{5}}}{5}=\frac{(0,10,0)-(0,8,4)}{5\sqrt{5}}=\frac{1}{5\sqrt{5}}(0,2,-4)=\frac{2}{5\sqrt{5}}(0,1,-2).$$

So, $(0,1,-2)$ is certainly parallel to $\mathbf{T}'(t)$ and so it is also parallel to $\mathbf{N}(t)$. So the cross product of that vector and $(0,2,1)$ will give us a vector that is parallel to the bi-normal vector, and so a vector that is perpendicular to the osculating plane. We compute $(0,1,-2)\times(0,2,1)=(5,0,0)$. So $(5,0,0)$ is a vector we can use and an equation of the osculating plane is $5x=0$.

(b) The normal plane is perpendicular to the $\mathbf{r}'(t)=(0,2t,1)$ while the given plane is perpendicular to the vector $(0,6,3)$. So, the planes are parallel if these two vectors are parallel. Two vectors are parallel if one is a multiple of the other. So, we check when $(0,2t,1)=k(0,6,3)$ for some number *k*. Equating the components and solving we get $k=\dfrac{1}{3}$ and $t=1$. So, the point we were looking for happens when $t=1$ and it is $(1,1,1)$.

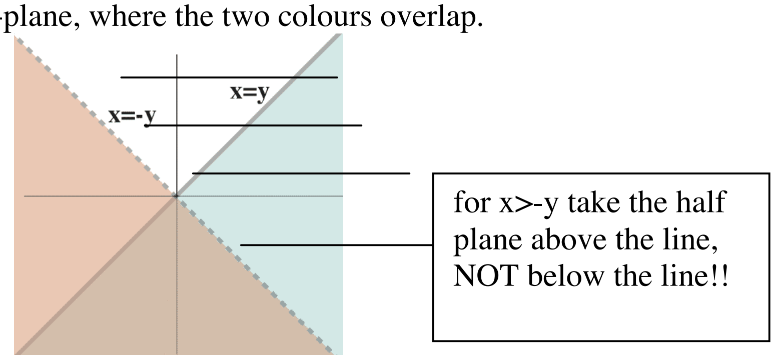
3. [6 marks]

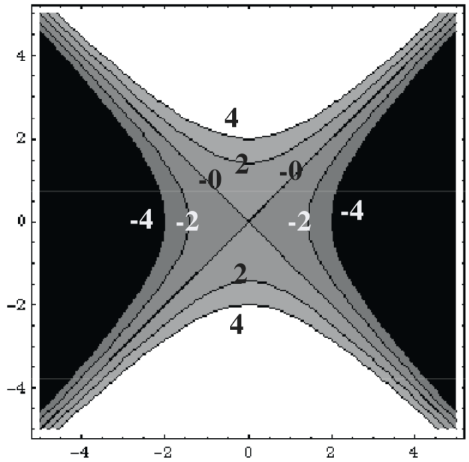
[2] (a) Find and sketch the domain of the function $f(x,y)=\sqrt{x-y}\,\ln(x+y)$.

[2] (b) Sketch the graph of the function $f(x,y)=y^2-x^2$. You may use computers or the webMathematica page.

[2] (c) Sketch a contour map (also know as a topographic map) of the function $f(x,y)=y^2-x^2$ by showing at least 4 level curves.

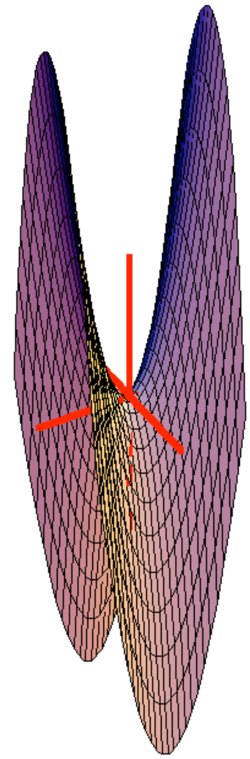
Solution. (a) The root and the logarithm tell us that $x-y\ge 0$ and $x+y>0$ respectively. Tidy this a bit to get $x\ge y$ and $x>-y$. So the domain of the function consists of all pairs (x,y) satisfying these two inequalities. A picture of the domain is given below: it consists of the lower quarter-plane, where the two colours overlap.





(b) A sketch of the graph of the function is given to the right.

(c) The contour plot is given to the left. We show 5 contours obtained from the traces z=-4, z=-2, z=0, z=2 and z=4



4. [6 marks].

[3] (a) Use the definition of limit to show that $\lim_{(x,y)\to(0,0)} 2xy = 0$. (Hint: you may need the observation that $(x-y)^2 \geq 0$.)

[3] (b) Show that the following limit does not exist: $\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^4+y^4}$.

Solution. (a) We want to show that for every $\varepsilon > 0$ there is a $\delta > 0$ (that depends on ε) such that if $\sqrt{x^2+y^2} < \delta$ then $|2xy| < \varepsilon$. Start with the hint and apply it to $(|x|-|y|)^2$ to get $|x|^2 - 2|xy| + |y|^2 \geq 0$, i.e., $|x^2| + |y^2| \geq 2|xy|$. Since the squares are never negative it follows that $x^2 + y^2 \geq |2xy|$. Now it is easy: choose $\delta = \sqrt{\varepsilon}$. Then if $\sqrt{x^2+y^2} < \delta$ we get $\sqrt{x^2+y^2} < \sqrt{\varepsilon}$ and so $x^2 + y^2 < \varepsilon$. But since $|2xy| \leq x^2 + y^2$, we have $|2xy| < \varepsilon$, which is what wanted.

(b) Along $x = 0$ we get $\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^4+y^4} = \lim_{y\to 0} \frac{2(0^2)y^2}{0^4+y^4} = 0$, while along $y = x$ we find $\lim_{x\to 0} \frac{2x^2x^2}{x^4+x^4} = \lim_{x\to 0} \frac{2x^4}{2x^4} = 1$. Since these two limits are distinct, the original limit does not exist.