136.270 Solutions

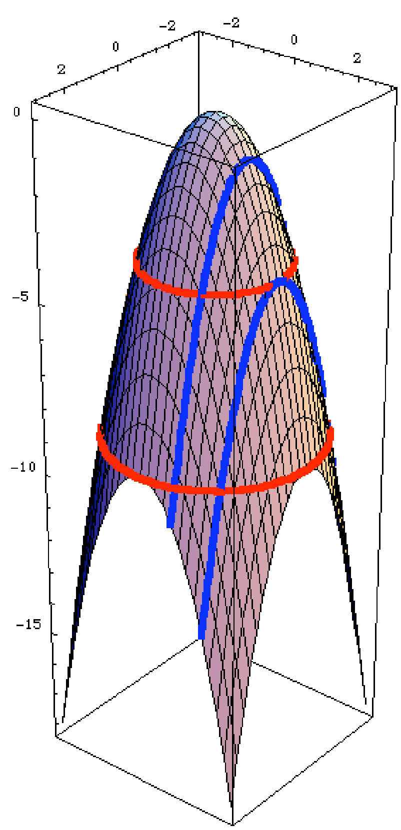
Assignment 1 (Sections 13.1-13.7, 14.1-14.2)

Handed: October 4, 2004. Due: October 13, 2004 in

1. [6 marks] Find parametric equations of the line through the point (0,1,2), that is perpendicular to the line $x = 1+t,\ \ y = 1-t,\ \ z = 2t$ and intersects that line.

Solution. The vector $\mathbf{v} = (1,-1,2)$ is parallel to the given line. So, it is perpendicular to the plane through the given point and perpendicular to the given line. The equation of that plane is $(1,-1,2)\bullet((x,y,z)-(0,1,2)) = 0$ which, after some simplification, becomes $x - y + 2z = 3$. The line we want lies in that plane. Moreover, it is the line passing through the given point (0,1,2) and the point of intersection of the plane we have just found and the given line. We find the latter by solving $x = 1+t,\ \ y = 1-t,\ \ z = 2t$ and $x - y + 2z = 3$. Substituting the first three into the fourth gives $(1+t)-(1-t)+2(2t) = 3$, i.e. $6t = 3$ and so $t = \frac{1}{2}$, which in turn tells us that $x = \frac{3}{2},\ \ y = \frac{1}{2}\ \ z = 1$.

So, we need to find the line through (0,1,2) and $\left(\frac{3}{2},\frac{1}{2},1\right)$. The vector $\left(\frac{3}{2},\frac{1}{2},1\right)-(0,1,2) = \left(\frac{3}{2},-\frac{1}{2},-1\right)$ is parallel to that line. So, the equation of the line we want is $x = 0+\frac{3}{2}t,\ \ y = 1-\frac{1}{2}t,\ \ z = 2-t$.



2. [5 marks] [3] (a) Sketch the surface $x^2 + y^2 = -z$. Sketch at least two traces of that surface with planes of type $y = c$ (c various constants) and at least two traces with planes of type $z = c$ (c various constant). (You may, if you want, use computers to do this; in particular, you may use the webMathematica page for this course.)

[2] (b) Find parametric equations of the curve in the intersection of the surface in (a) and the plane $-x - y - z = 3$.

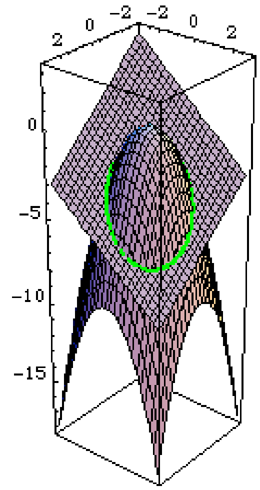
Solution. (a)
The picture to the left shows the surface together with the traces with z=-4 and z=-9 (in red) and y=1 and y=2 (in blue).

(b) Set (say) $x = t$. Now solve for y and z in terms of t: get

$$x = t, y = \frac{1}{2}(1+\sqrt{13+4t-4t^2}), z = \frac{1}{2}(-7-2t-\sqrt{13+4t-4t^2})$$

for one half of the curve, and

$x = t, y = \frac{1}{2}(1 - \sqrt{13 + 4t - 4t^2}), z = \frac{1}{2}(-7 - 2t + \sqrt{13 + 4t - 4t^2})$
for the other half of the curve, The two surfaces as well as their intersection curve (in green) are shown to the right.



3. [7 marks]

[1.5] (a) Find the rectangular coordinates of the point $(r, \theta, z) = \left(5, \frac{\pi}{6}, 6\right)$ given in cylindrical coordinates.

[1.5] (b) Find the rectangular coordinates of the point $(\rho, \theta, \phi) = \left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ given in spherical coordinates.

[1.5] (c) Find the cylindrical coordinates of the point $(x, y, z) = (3, 4, 5)$ given in rectangular coordinates.

[1.5] (d) Find the spherical coordinates of the point $(x, y, z) = \left(1, 1, \sqrt{2}\right)$ given in rectangular coordinates.

[1] Plot all of the above points in a single coordinate system.

Solution. (a) We are given that $r = 5,\ \theta = \frac{\pi}{6},\ z = 6$. So, $x = r\cos\theta = 5\cos\frac{\pi}{6} = 5\frac{\sqrt{3}}{2}$, $y = r\cos\theta = 5\sin\frac{\pi}{6} = \frac{5}{2}$, z=6. So, the rectangular coordinates of the point are $(5\frac{\sqrt{3}}{2}, \frac{5}{2}, 6)$.

(b) Substitute in $x = \rho\sin\varphi\cos\theta, x = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$ to get $x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}, y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6}}{2},\ z = 2\cos\frac{\pi}{3} = 1$, so the rectangular coordinates of the point are $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)$.

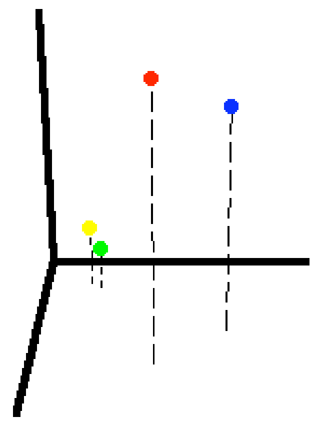
(c) Since $r = \sqrt{x^2 + y^2}$ we compute that r=5. The polar angle satisfies $\tan\theta = \frac{y}{x} = \frac{4}{3}$ and so $\theta = \tan^{-1}\frac{4}{3}$. So, the cylindrical coordinates are $(r, \theta, z) = \left(5, \tan^{-1}\frac{4}{3}, 5\right)$.

(d) Note first that the point is in the first octant; so both φ and θ are between 0 and $\frac{\pi}{2}$. Compute: $\rho=\sqrt{x^2+y^2+z^2}=\sqrt{4}=2$, $\varphi=\cos^{-1}\frac{\sqrt{2}}{2}=\frac{\pi}{4}$ and

$\sin\theta=\frac{x}{\rho\sin\varphi}=\frac{1}{2\frac{\sqrt{2}}{2}}=\frac{1}{\sqrt{2}}$ so that

$\theta=\sin^{-1}\frac{\sqrt{2}}{2}=\frac{\pi}{4}$. So, the spherical coordinates are $(\rho,\theta,\phi)=\left(2,\frac{\pi}{4},\frac{\pi}{4}\right)$.

The plot of all 4 points is given to the right ((a) is in red, (b) is in green, (c) is in blue and (d) is in yellow; the dashed lines just so that we can see better where the points are in 3 dimensions).



4. [7 marks].

[2] (a) For $\vec{\mathbf{r}}(t)=\left(e^{-t},\frac{t-1}{t+1},\tan^{-1}t\right)$ find $\lim_{t\to\infty}\vec{\mathbf{r}}(t)$.

[3] (b) Find all points on the curve $\vec{\mathbf{r}}(t)=\left(1-t^2,4t^2-1,t^4-8t^2\right)$ where that curve has tangent lines parallel to the vector $\mathbf{v}=(-2,-8,12)$.

[2] (c) Find the unit tangent vector of the curve $\vec{\mathbf{r}}(t)=\left(3t+1,4t^2-1,t^4-8t^2\right)$ at the point $(1,-1,0)$.

Solution. (a) $\lim_{t\to\infty}\vec{\mathbf{r}}(t)=\lim_{t\to\infty}\left(e^{-t},\frac{t-1}{t+1},\tan^{-1}t\right)=\left(\lim_{t\to\infty}e^{-t},\lim_{t\to\infty}\frac{t-1}{t+1},\lim_{t\to\infty}\tan^{-1}t\right)=(0,1,\frac{\pi}{2})$.

(b) $\vec{\mathbf{r}}'(t)=\left(-2t,8t,4t^3-16t\right)$. We want the point where this vector is parallel to **v**, i.e., where $\left(-2t,8t,4t^3-16t\right)=k(-2,-8,12)$ for some non-zero number *k*. The first coordinates tell us that *t*=*k*, the second coordinate yields *t*=-*k*, from where it follows that *k*=0. So, the curve is such that in no point the tangent vector is parallel to **v**.

(c) $\vec{\mathbf{r}}'(t)=\left(3,8t,4t^3-16t\right)$. The point $(1,-1,0)$ happens when *t*=0, and so at that moment we have $\vec{\mathbf{r}}'(1)=\left(3,0,0\right)$. The length of this vector is obviously $\left|\vec{\mathbf{r}}'(0)\right|=3$. So, the unit tangent vector wanted is (1,0,0).