136.270 Multivariable Calculus SOLUTIONS
Midterm Exam
October 27 2003
(5:30-6:30, 100 St Paul)

(If you need more space use the back side and indicate that you have done so.)

1. We are given the vector-valued function $\mathbf{r}(t) = (e^t, \sqrt{2}t, e^{-t})$ (In order to avoid bad initial errors that may lead to difficult computation, I emphasize that in the second coordinate t is outside the root).

(a) Compute the unit tangent vector **T** at the moment when $t = 0$.

Solution: $\mathbf{r}'(t) = (e^t, \sqrt{2}, -e^{-t})$ and when $t = 0$ we get $\mathbf{r}'(0) = (1, \sqrt{2}, -1)$. The length of that vector is $|\mathbf{r}'(0)| = \sqrt{1^2 + (\sqrt{2})^2 + (-1)^2} = \sqrt{4} = 2$. So, the unit tangent vector at that moment is $\mathbf{T}(0) = \frac{1}{2}(1, \sqrt{2}, -1)$.

(b) Find the arc-length of that curve for $0 \le t \le 5$.

Solution. $|\mathbf{r}'(t)| = |(e^t, \sqrt{2}, -e^{-t})| = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$. So, the arc length we want is $\int_0^5 |\mathbf{r}'(t)| dt = \int_0^5 (e^t + e^{-t}) dt = e^t - e^{-t}\Big|_0^5 = e^5 - e^{-5}$.

2. Evaluate the indicated limit or show it does not exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^2}$

Solution. Switch to polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)^2}{(r\cos\theta)^2 + (r\sin\theta)^2} = \lim_{r\to 0} \frac{r^3\cos\theta\sin\theta}{r^2} = \lim_{r\to 0} r\cos\theta\sin\theta = 0.$$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{2x^4 + y^4}$

Solution. Along the curve x=0 the limit is obviously 0. Along the curve $y=x$ we get $\lim\limits_{\substack{(x,y)\to(0,0)\\ along\ \ y=x}} \dfrac{x^2y^2}{2x^4+y^4} = \lim\limits_{x\to 0} \dfrac{x^2x^2}{2x^4+x^4} = \lim\limits_{x\to 0} \dfrac{x^4}{3x^4} = \dfrac{1}{3}$. Since we got two different limits, it follows that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{2x^4+y^4}$ does not exist.

(c) $\lim\limits_{(x,y)\to(1,0)} \dfrac{e^y - 1}{2x + y - 2}$

Solution. Along x=1 we have:

$$\lim_{\substack{(x,y)\to(1,0)\\ along\ \ x=1}} \frac{e^y-1}{2x+y-2} = \lim_{y\to 0} \frac{e^y-1}{2+y-2} = \lim_{y\to 0} \frac{e^y-1}{y} \overset{Using\ \ L'Hopital}{=} \lim_{y\to 0} e^y = 1.$$

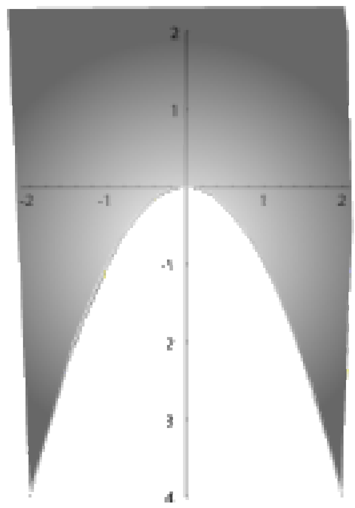
On the other hand, along y=0 we have $\lim\limits_{\substack{(x,y)\to(1,0)\\ along\ \ y=0}} \dfrac{e^y-1}{2x+y-2} = \lim\limits_{x\to 1} \dfrac{1-1}{2x+0-2} = \lim\limits_{x\to 1} \dfrac{0}{2x-2} = 0$.

So, the limit does not exist.

3. Consider the function $f(x,y) = \dfrac{x-y}{\sqrt{x^2+y}}$.

(a) Sketch (in the x-y plane) the domain of that function.

Solution. $f(x,y) = \dfrac{x-y}{\sqrt{x^2+y}}$ is defined only when $x^2 + y > 0$. The boundary of the region of all points satisfying that inequality is the curve $x^2 + y = 0$, i.e. $y = -x^2$. We sketch that curve below. It divides the plane into two parts. By testing (the coordinates of) any point in the plane outside that curve we conclude that the domain is the set of all points in the shaded region (excluding the boundary).



(b) Find the equation of the tangent plane to the graph of that function at the point when x=1 and y=1.

Solution. The equation of the tangent plane at any point (x_0, y_0) is $z = f(x_0,y_0) + \dfrac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \dfrac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$. So, we need the partial derivatives at the point $(1,1)$. We compute $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\sqrt{x^2+y} - (x-y)\dfrac{2x}{2\sqrt{x^2+y}}}{x^2+y}$, and so $\dfrac{\partial f}{\partial x}(1,1) = \dfrac{\sqrt{2}}{2}$.

Similarly $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{-\sqrt{x^2+y} - (x-y)\dfrac{1}{2\sqrt{x^2+y}}}{x^2+y}$, and so $\dfrac{\partial f}{\partial y}(1,1) = -\dfrac{\sqrt{2}}{2}$. Finally we find from the given function that $f(1,1) = 0$. So, the equation of the tangent plane is $z = +\dfrac{\sqrt{2}}{2}(x-1) - \dfrac{\sqrt{2}}{2}(y-1)$.

4. **(a)** Use the chain rule to find $\dfrac{\partial f}{\partial u}$ if $f(x,y) = e^y + 3y\sin x$, $x = \ln(2u+3v)$, $y = \dfrac{v}{u}$.

Solution.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 3y\cos x\frac{2}{2u+3v} + (e^y + 3\sin x)\left(-\frac{v}{u^2}\right) =$$
$$= 3\frac{v}{u}\cos(\ln(2u+3v))\frac{2}{2u+3v} + (e^{\frac{v}{u}} + 3\sin(\ln(2u+3v)))\left(-\frac{v}{u^2}\right)$$

(b) The equation $xz^3 + 2yz - 3xy = 0$ defines *z* as a function on *x* and *y*. Compute $\dfrac{\partial z}{\partial x}$ at the point when *x*=1, *y* = 1 and *z*=1.

Solution. Denote $F(x,y,z)=xz^3+2yz-3xy$. Then $\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{z^3-3y}{3xz^2+2y}$. At the given point we have $\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{1^3-3(1)}{3(1)(1)^2+2(1)}=\frac{2}{5}$.

5. **(a)** State the definition of $\lim_{(x,y)\to(a,b)} f(x,y)$. That is, what does it mean to say that $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to some number L?

Solution. $\lim_{(x,y)\to(a,b)} f(x,y)=L$ means that for every $\varepsilon>0$ there exists $\delta>0$ such that if $\sqrt{(x-a)^2+(y-b)^2}<\delta$ then $|f(x,y)-L|<\varepsilon$.

(b) Show **using the definition of *limit*** that $\lim_{(x,y)\to(0,0)} xy=0$. No points will be given if other methods are used.

Solution. We want to show that for every $\varepsilon>0$ there exists $\delta>0$ such that if $\sqrt{(x-0)^2+(y-0)^2}<\delta$ then $|xy-0|<\varepsilon$. After simplifying a bit, this means that we want to show that for every $\varepsilon>0$ there exists $\delta>0$ such that if $\sqrt{x^2+y^2}<\delta$ then $|xy|<\varepsilon$. So, start with any $\varepsilon>0$ and take $\delta=\sqrt{\varepsilon}$ and suppose $\sqrt{x^2+y^2}<\delta$. Then $\sqrt{x^2+y^2}<\sqrt{\varepsilon}$, and so $x^2+y^2<\varepsilon$. But if the last inequality is true, then, since $x^2\le x^2+y^2$ we have that $x^2<\varepsilon$, or $|x|<\sqrt{\varepsilon}$. Similarly (by symmetry) we have that $|y|<\sqrt{\varepsilon}$ too. With these two inequalities we find that $|xy|=|x||y|<\sqrt{\varepsilon}\sqrt{\varepsilon}=\varepsilon$ as needed.