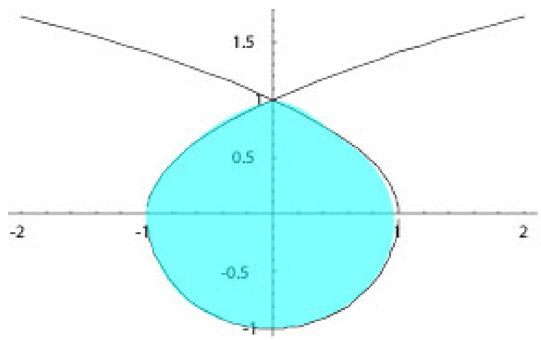
136.270 Solutions

Assignment 4 (Sections 16.1-16.4)

1. [8 marks] Evaluate $\iint_D (4xy^3 - 4x^2y)dA$ where D is the region bounded by $y = -\sqrt{1-x^2}$, $y = \sqrt{1-x}$ and $y = \sqrt{1+x}$. Sketch D.

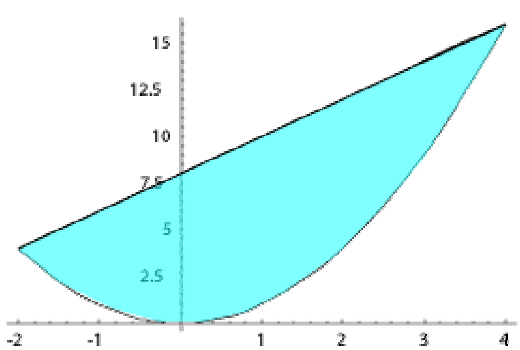
Solution. The parabolas $y = \sqrt{1-x}$ and $y = \sqrt{1+x}$ intersect at the point (0,1) (we get by solving the system of the two equations). The region D is shown in the picture below. We can describe it as follows: $-1 \le x \le 0$ and $-\sqrt{1-x^2} \le y \le \sqrt{1+x}$ for the part of D to the left of the y-axis, and $0 \le x \le 1$ and $-\sqrt{1-x^2} \le y \le \sqrt{1-x}$ for the part of D to the right of the y-axis.

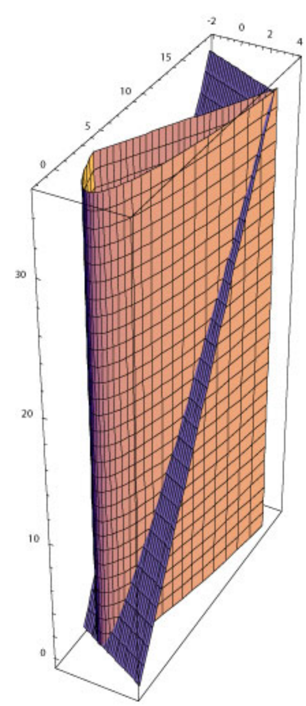


So we have $\iint_D (4xy^3 - 4x^2y)dA = \int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{\sqrt{1+x}} (4xy^3 - 4x^2y)dydx + \int_{0}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x}} (4xy^3 - 4x^2y)dydx =$
= (simple computations in between) = 1/5

2. [8 marks] Find the volume V of the solid S bounded by the xy-plane, the cylinder $y = x^2$, and the planes $z = x + 2y$ and $y = 2x + 8$. Sketch S.

The solid S is show in the (3D) picture to the right. It is bounded above by the plane $z = x + 2y$ and below by the region B bounded by the curves $y = x^2$ and $y = 2x + 8$ in the xy-plane (shown in the picture below).

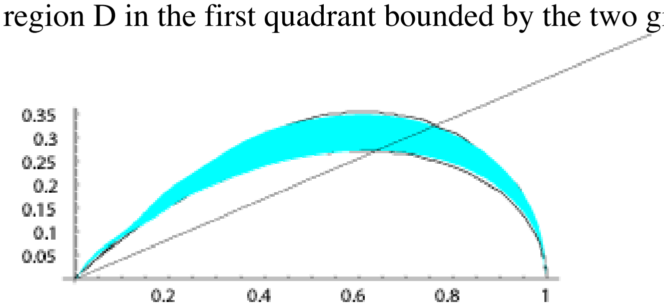




In the intersection of the two curves bounded the above region we have (after solving the system $y = x^2$, $y = 2x + 8$) that x=-2 (for the left hand side point) and x=4 (for the right hand side point). So, the volume we want is $\iint_B (x + 2y)dA = \int_{-2}^{4}\int_{x^2}^{2x+8}(x + 2y)dydx$ = (after some easy computation) 2484/5.

3. [8 marks] Use double integrals and polar coordinates to find the area **in the first quadrant** between the lemniscate $r^2 = \cos 2\theta$ and the four-leaf rose $r = \cos 2\theta$.

First we sketch the region D in the first quadrant bounded by the two given curves:



A small analysis shows that we get this region as θ changes from 0 to $\frac{\pi}{4}$. The semiline that we show is there only to help us see that for a fixed angle θ the other coordinate r changes from the curve $r = \cos 2\theta$ (the curve on the boundary of the region D that is closer to the origin) to the curve $r = \sqrt{\cos 2\theta}$ (farther from the origin). Consequently, the area of D is $\int_0^{\pi/4}\int_{\cos(2\theta)}^{\sqrt{\cos(2\theta)}} rdr \ \ d\theta$. Easy computation yields $\int_0^{\pi/4}\int_{\cos(2\theta)}^{\sqrt{\cos(2\theta)}} rdr \ \ d\theta = \frac{1}{4} - \frac{\pi}{16}$.

1 mark for free (to celebrate the end of the term)