136.270

Assignment 3 Brief Solutions

1. [5 marks]

(a) Find the directional derivative of the function $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ in the direction of the unit vector $\mathbf{u}=\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$ at the point (1,-2).

(b) Find the directions and the values of the smallest and the largest directional derivatives of the function $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ at the point (1,-2).

Solutions. (a) $\nabla f(x,y)=(\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y})=\left(\dfrac{4xy^2}{(x^2+y^2)^2},-\dfrac{4yx^2}{(x^2+y^2)^2}\right)$ and $\nabla f(1,-2)=\left(\dfrac{16}{25},\dfrac{8}{25}\right)$.
So $D_u f(1,-2)=\nabla f(1,-2)\bullet u=\left(\dfrac{16}{25},\dfrac{8}{25}\right)\bullet\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)=\dfrac{16}{25}\dfrac{1}{\sqrt{2}}+\dfrac{8}{25}\dfrac{1}{\sqrt{2}}$.

(b) The largest value of the directional derivative at the given point is $|\nabla f(1,-2)|$, and that happens to be (after simplifying) $\dfrac{8}{25}\sqrt{5}$. The largest value is attained in the direction of the unit vector parallel to (and in the same direction as) $\nabla f(1,-2)$, which is (after simplifying) $\dfrac{1}{|\nabla f(1,-2)|}\nabla f(1,-2)=\dfrac{1}{\sqrt{5}}(2,1)$.

The smallest value of the directional derivative at the given point is $-|\nabla f(1,-2)|$, which is $-\dfrac{8}{25}\sqrt{5}$. The smallest value is attained in the direction of the unit vector $-\dfrac{1}{\sqrt{5}}(2,1)$.

2. [5 marks] First locate the local extrema of the function $g(x,y)=\dfrac{x+y}{x^2+y^2+8}$, and then use the second derivative test to classify these local extrema (as local minima, local maxima or neither).

Solution. $\dfrac{\partial g}{\partial x}=\dfrac{y^2-x^2+8-2xy}{\left(x^2+y^2+8\right)^2}$ and $\dfrac{\partial g}{\partial y}=\dfrac{x^2-y^2+8-2xy}{\left(x^2+y^2+8\right)^2}$. For critical points we solve $\dfrac{\partial g}{\partial x}=0=\dfrac{\partial g}{\partial y}$, which yields the system $y^2-x^2+8-2xy=0$, $x^2-y^2+8-2xy=0$. Add these two equations to get $16=4xy$ or $4=xy$. Put this in any of the original equations to get $x^2 = y^2$ which means that $x = y$ or that $x = -y$. From this point it is easy to see that the solutions are (2,2) and (-2,-2).

We now compute $\frac{\partial^2 g}{\partial x^2}$, $\frac{\partial^2 g}{\partial y^2}$ and $\frac{\partial^2 g}{\partial x \partial y}$, and evaluate them at the two critical points.

For (-2,-2) we get $\left.\frac{\partial^2 g}{\partial x^2}\right|_{(-2,-2)} = \frac{1}{32}$, $\left.\frac{\partial^2 g}{\partial y^2}\right|_{(-2,-2)} = \frac{1}{32}$ and $\left.\frac{\partial^2 g}{\partial x \partial y}\right|_{(-2,-2)} = 0$, so that $D = \frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial x^2} - \left(\frac{\partial^2 g}{\partial x \partial y}\right)^2$ is $\frac{1}{1024}$ at that point. Since D is positive and $\left.\frac{\partial^2 g}{\partial x^2}\right|_{(-2,-2)}$ is also positive, it follows that we have a local minimum at (-2,-2).

For (2,2) we get $\left.\frac{\partial^2 g}{\partial x^2}\right|_{(2,2)} = -\frac{1}{32}$, $\left.\frac{\partial^2 g}{\partial y^2}\right|_{(2,2)} = -\frac{1}{32}$ and $\left.\frac{\partial^2 g}{\partial x \partial y}\right|_{(2,2)} = 0$, so that $D = \frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial x^2} - \left(\frac{\partial^2 g}{\partial x \partial y}\right)^2$ is $\frac{1}{1024}$ at that point. Since D is positive and $\left.\frac{\partial^2 g}{\partial x^2}\right|_{(-2,-2)}$ is negative, it follows that we have a local maximum at (2,2).

3. Consider the function $f(x,y) = x^2 - x - y + y^2$ over the points in the closed disk bounded by the circle $x = 2\cos t$, $y = 2\sin t$. Find and classify the **absolute** extrema of the function $f(x,y)$ over the given domain.

Solution. Compute $\frac{\partial f}{\partial x} = 2x+1$ and $\frac{\partial f}{\partial y} = 2y+1$. Solving $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ yields $\left(-\frac{1}{2},\frac{1}{2}\right)$ as the only critical points. We now use the second derivative test: $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(-1/2,1/2)} = 2$, $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(-1/2,1/2)} = 2$ and $\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(-1/2,1/2)} = 0$, so that $D = 4$. Since D>0 and $\frac{\partial^2 f}{\partial x^2} > 0$ at the critical point, it follows that the function has a local minimum at $\left(-\frac{1}{2},\frac{1}{2}\right)$.

Now we see what happens over the boundary. The function *f* over the boundary circle reduces to
$f(x,y) = x^2 - x - y + y^2 = 4\cos^2 t + 2\cos t - 2\sin t + 4\sin^2 t = 2\cos t - 2\sin t + 4 = g(t)$ (a function on t, where t changes from 0 to 2π). Then $g'(t) = -2\sin t - 2\cos t$ and this is 0 for $t = \frac{3\pi}{4}$ or $t = \frac{7\pi}{4}$.

We now compute the values of f(x,y)=g(t) over these two points, as well as the value of f over the point $\left(-\frac{1}{2},\frac{1}{2}\right)$ found above: $g\left(\frac{3\pi}{4}\right) = 4 - 2\sqrt{2}$, $g\left(\frac{7\pi}{4}\right) = 4 + 2\sqrt{2}$ and $f\left(-\frac{1}{2},\frac{1}{2}\right) = -\frac{1}{2}$. Comparing these we see that we have an absolute maximum of $4 + 2\sqrt{2}$

happening when $t = \frac{7\pi}{4}$, that is, when $x = 2\cos\frac{7\pi}{4}$ and $y = 2\sin\frac{7\pi}{4}$, and the absolute minimum of $-\frac{1}{2}$ happening at $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

4. [5 marks]. A silo is constructed with cylindrical walls and a conical roof. Find the dimensions of such a silo (the radius of the cylinder, the height of the cylinder and the height of the conic roof) of volume 10 000 cubic meters which has the smallest surface area. (You may need to browse through some of your (old) books for some formulas for volume and surface area of cylinders and cones.)

Solution. Denote by x the radius of the cylinder, by y the height of the cylinder and by z the height of the cone. The surface area S of the silo is the surface area of the cylinder plus the surface area of the cone, that is $s = 2x\pi y + x\pi\sqrt{x^2 + z^2}$. The volume of the silo is $v = x^2\pi y + \frac{x^2\pi z}{3}$ and we are given that this is 10000, so that $x^2\pi y + \frac{x^2\pi z}{3} = 10000$. So the problem reduces to the following: find the absolute minimum of the function $s(x,y,z) = 2x\pi y + x\pi\sqrt{x^2 + z^2}$ under the condition that $x^2\pi y + \frac{x^2\pi z}{3} = 10000$. We will use Lagrange multipliers. Denote $F(x,y,z,\lambda) = 2x\pi y + x\pi\sqrt{x^2 + z^2} + \lambda(x^2\pi y + \frac{x^2\pi z}{3} - 10000)$ and solve $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ and $\frac{\partial F}{\partial \lambda} = 0$. After computing the partial derivatives the system becomes $2\pi y + \pi\sqrt{x^2 + z^2} + \frac{x^2\pi}{\sqrt{x^2 + z^2}} + 2\lambda x\pi y + \frac{2\lambda x\pi z}{3} = 0$, $2x\pi + \lambda x^2\pi = 0$, $\frac{x\pi z}{\sqrt{x^2 + z^2}} + \lambda\frac{x^2\pi}{3} = 0$ and $x^2\pi y + \frac{x^2\pi z}{3} = 10000$**.** The only solutions giving positive values for the dimensions is $x = 10\sqrt[6]{5}\sqrt[3]{\frac{6}{\pi}}$, $y = 2\sqrt[3]{5^2}\sqrt[3]{\frac{6}{\pi}}$ and $x = 4\sqrt[3]{5^2}\sqrt[3]{\frac{6}{\pi}}$. It is obvious from the "nature of the problem" that an absolute minimum exists (or else we could make the surface area as small as we wish by choosing some values for x, y and z). So, the above must be the point where that absolute minimum is attained.

5. [5 marks] Use the method of Lagrange multipliers to find and classify the extrema of the function $f(x,y) = xy$ subject to the constraint $x^2 + y^2 - 4 = 0$.

Solution. Denote $F(x,y,\lambda) = f(x,y) + \lambda(x^2 + y^2 - 4) = xy + \lambda(x^2 + y^2 - 4)$. The equations $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial \lambda} = 0$ are $y + 2\lambda x = 0$, $x + 2\lambda y = 0$ and $x^2 + y^2 - 4 = 0$. Solving this gives 4 solutions for x and y: $(-\sqrt{2}, -\sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$ and

$(\sqrt{2},\sqrt{2})$. We evaluate $f(x,y)=xy$ at these 4 points to get $f(-\sqrt{2},-\sqrt{2})=2$, $f(\sqrt{2},-\sqrt{2})=-2$, $f(-\sqrt{2},\sqrt{2})=-2$ and $f(\sqrt{2},\sqrt{2})=2$. So, *f* attains its absolute minimum of -2 at $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$, while it attains its absolute maximum of 2 at $(-\sqrt{2},-\sqrt{2})$ and $(\sqrt{2},\sqrt{2})$. [This is a consequence of the theorem regarding absolute extrema of functions over closed bounded domains, which is the case with the set of all points on the circle $x^2+y^2-4=0$.]