136.270

Assignment 2 (Sections 14.3, 15.1-15.3)

Handed: Oct.10 2003. Due: Oct.17 2003 in class. Show your work; providing answers without justifying them would not be sufficient.

1. [8 marks] A spiral curve is defined by the vector function $\vec{r}(t) = (4\cos t, 4\sin t, 3t)$.
   (a) Find the arc length function $s(t)$ measured from the point (4,0,0).
   (b) Reparametrize the curve in terms of the arc length function s measured from the point (4,0,0).
   (c) Compute the curvature of that spiral curve at any moment in terms of s.
   (d) Compute the curvature in terms of t directly from $\vec{r}(t) = (2\cos t, 2\sin t, 3t)$.
   (e) Find the equations of the normal and the osculating plane to the spiral at the point (4,0,0).

Solution.

[1.5] (a) $\vec{r}\,'(t) = (-4\sin t, 4\cos t, 3)$ and so

$\left|\vec{r}\,'(t)\right| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = \sqrt{25} = 5$. Consequently, $s(t) = \int_0^t 5\,du = 5t$. Note that the lower limit of the integral is t=0 since at that moment we get the point (4,0,0).

[1.5] (b) If follows from (a) that $t = \frac{s}{5}$, so that $\vec{r}(s) = (4\cos\frac{s}{5}, 4\sin\frac{s}{5}, 3\frac{s}{5})$.

[1.5] (c) $\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|$ where $\mathbf{T}$ is the unit tangent vector. So

$$\mathbf{T}(s) = \frac{\vec{r}\,'(s)}{\left|\vec{r}\,'(s)\right|} = \frac{\left(-\frac{4}{5}\sin\frac{s}{5}, \frac{4}{5}\cos\frac{s}{5}, \frac{3}{5}\right)}{1} = \left(-\frac{4}{5}\sin\frac{s}{5}, \frac{4}{5}\cos\frac{s}{5}, \frac{3}{5}\right).$$ So

$$\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\left(-\frac{4}{25}\cos\frac{s}{5}, -\frac{4}{25}\sin\frac{s}{5}, 0\right)\right| = \frac{4}{25}.$$

[1.5] (d) NOTE: I have used the intended $\vec{r}(t) = (4\cos t, 4\sin t, 3t)$ rather than $\vec{r}(t) = (2\cos t, 2\sin t, 3t)$ (some of you have been told of the typo). You get all of the marks in both cases.

$\kappa(t) = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right|$. We have already computer that $\left|\vec{r}\,'(t)\right| = 5$ and that $\vec{r}\,'(t) = (-4\sin t, 4\cos t, 3)$.

So, $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\left|\mathbf{r}'(t)\right|} = \frac{(-4\sin t, 4\cos t, 3)}{5}$. So $\mathbf{T}'(t) = \frac{(-4\cos t, -4\sin t, 0)}{5}$. Consequently

$\left|\mathbf{T}'(t)\right| = \frac{\left|(-4\cos t, -4\sin t, 0)\right|}{5} = \frac{4}{5}$. Finally $\kappa(t) = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| = \frac{4}{5}\frac{1}{5} = \frac{4}{25}$ as we have already found above.

[2] (e) $\vec{r}'(t) = (-4\sin t, 4\cos t, 3)$ is tangent to the curve and normal to the normal plane. At *t=0* (the moment we get the point (4,0,0)) we compute that $\vec{r}'(0) = (0,4,3)$. So, the equation of the normal plane through (4,0,0) is $0(x-4)+4(y-0)+3(z-0)=0$.

For the osculating plane we need the bi-normal vector, or, for that matter, any vector that is parallel to the bi-normal vector. The vector $\mathbf{T}(0)\times\mathbf{T}'(0)$ is such. Using what we have computed above we find that $\mathbf{T}(0) = \frac{1}{5}(0,4,3)$ and $\mathbf{T}'(0) = \frac{1}{5}(-4,0,0)$. The cross product of these two is $\frac{1}{25}(0,-12,16)$. So, the bi-normal plane has the equation $0(x-4) - \frac{12}{25}(y-0) + \frac{16}{25}(z-0) = 0$.

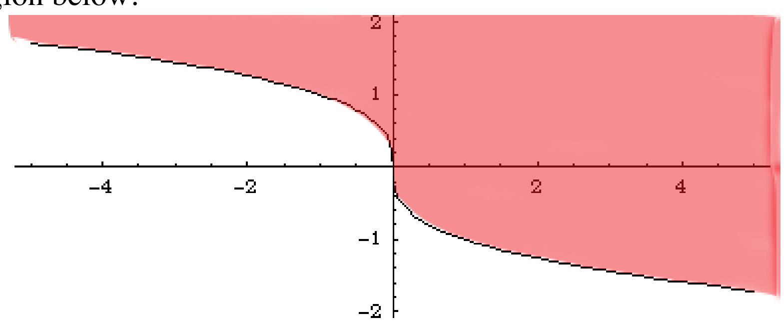
2. [4 marks] Determine **and sketch** (in the xy-plane) the domain of each of the following functions.

(a) $f(x,y) = \sqrt{x+y^3}$

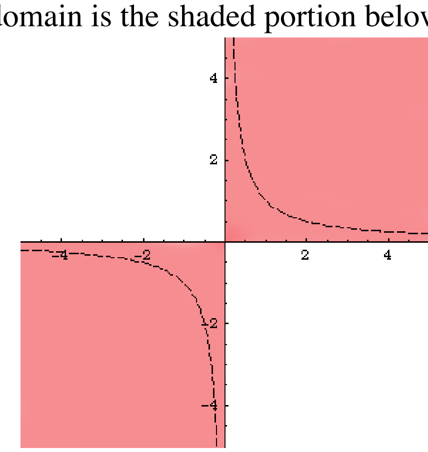
(b) $g(x,y) = \dfrac{x+y}{1-\sqrt{xy}}$

Solution.

[2] (a) We must have $x + y^3 \geq 0$ or $y^3 \geq -x$. The points satisfying that inequality are in the shaded region below.



[2] (b) $g(x,y) = \dfrac{x+y}{1-\sqrt{xy}}$ is well defined when $xy \geq 0$ and when $1-\sqrt{xy} \neq 0$. The first inequality happens for the points in the first and third quadrant (including the axes), while the second inequality happens for all points outside the curve $1-\sqrt{xy}=0$, that is, outside the hyperbola $xy = 1$. The domain is the shaded portion below, excluding the dotted lines.



[8] 3. [8 marks]

(a) Find the limit or show it does not exist: $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

(b) Find the limit or show it does not exist: $\lim_{(x,y)\to(1,-1)} \frac{x^2-1}{x^2+y^3}$

(c) Can the function $f(x,y)=\frac{x^3}{x^2+y^2}+1$ be defined at the point (0,0) so it becomes continuous? Do not forget to justify your answer.

Solution.

[2.5] (a) Switch to polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0}\frac{1-\cos(r^2)}{r^2} \overset{L'Hospital}{=} \lim_{r\to 0}\frac{-2r\sin(r^2)}{2r} = 0$$

[2.5] (b) Along the curve $y=-x$ (passing through (1,-1)) we have

$$\lim_{\substack{(x,y)\to(1,-1)\\ \text{along that curve}}} \frac{x^2-1}{x^2+y^3} = \lim_{x\to 1}\frac{x^2-1}{x^2+(-x)^3} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x^2(1-x)} = \lim_{x\to 1}\frac{-(x+1)}{x^2} = -\frac{1}{2}.$$

On the other hand along the curve $y=-1$ (which also passes through (1,-1)), we have:

$$\lim_{\substack{(x,y)\to(1,-1)\\ \text{along } y=-1}} \frac{x^2-1}{x^2+y^3} = \lim_{x\to 1}\frac{x^2-1}{x^2+(-1)^3} = \lim_{x\to 1}\frac{x^2-1}{x^2-1} = \lim_{x\to 1} 1 = 1.$$

Since we got two different answers when approaching (1,-1) along two different curves, we conclude that the original limit does not exist.

[3] (c) The function $f(x,y)=\frac{x^3}{x^2+y^2}+1$ will be continuous at (0,0) if we define it there so that $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. This is only possible if the limit exists. We compute:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}+1 \overset{polar}{=} \left(\lim_{r\to 0}\frac{(r\cos\theta)^3}{r^2}\right)+1 = (\lim_{r\to 0} r\cos^3\theta)+1 = 1.$$

So, by defining $f(0,0)=1$ the function will become continuous.

4. [4 marks].

[2] (a) Find $f_x(0,0)$ and find $f_y(x,y)$ if $f(x,y)=e^{xy}\sin(x+y+\pi)$.

[3] (b) Find all (four) second order partial derivatives of $g(x,y)=xy^2+\ln(x+y)$.

Solution.

[2] (a) $f_x(x,y) = ye^{xy}\sin(x+y+\pi)+e^{xy}\cos(x+y+\pi)$. So $f_x(0,0)=\cos(\pi)=-1$.

$f_y(x,y) = xe^{xy}\sin(x+y+\pi)+e^{xy}\cos(x+y+\pi)$.

[3] (b) $g_x(x,y) = y^2+\frac{1}{x+y}$, $g_y(x,y)=2xy+\frac{1}{x+y}$. So we have

$$g_{xx}(x,y) = -\frac{1}{(x+y)^2},\quad g_{xy}(x,y) = 2y-\frac{1}{(x+y)^2}$$

$$g_{yx}(x,y) = 2y-\frac{1}{(x+y)^2} \text{ and } g_{yy}(x,y) = 2x-\frac{1}{(x+y)^2}.$$