136.270

Assignment 1 (Sections 13.1-13.7, 14.1-14.2) Solutions

Handed: Sept. 24 2003. Due: Oct.1

1. [5 marks] Find the equation of the plane which contains the point $(3,-1,5)$ and is perpendicular to the intersection of the planes $x-5y+2z=3$ and $4x+y-z=2$.

Solution.

Before we start (?) let us identify the vectors n_1=(1,-5,2) and n_2 =(4,1,-1): the first one is normal (perpendicular) to the first given plane (and we extract it from the coefficients in that equation), while the second one is normal to the second given plane.

Observe first the following:

- The plane we want, being perpendicular to both of the given planes, must be perpendicular to their intersection line.
- Since every line in a plane is perpendicular to the normal vector of the plane, it follows that the line of intersection of the two planes is perpendicular to both of the normal vectors to the two given planes. Consequently, that intersection line must be parallel to the cross product $n_1 \times n_2$.

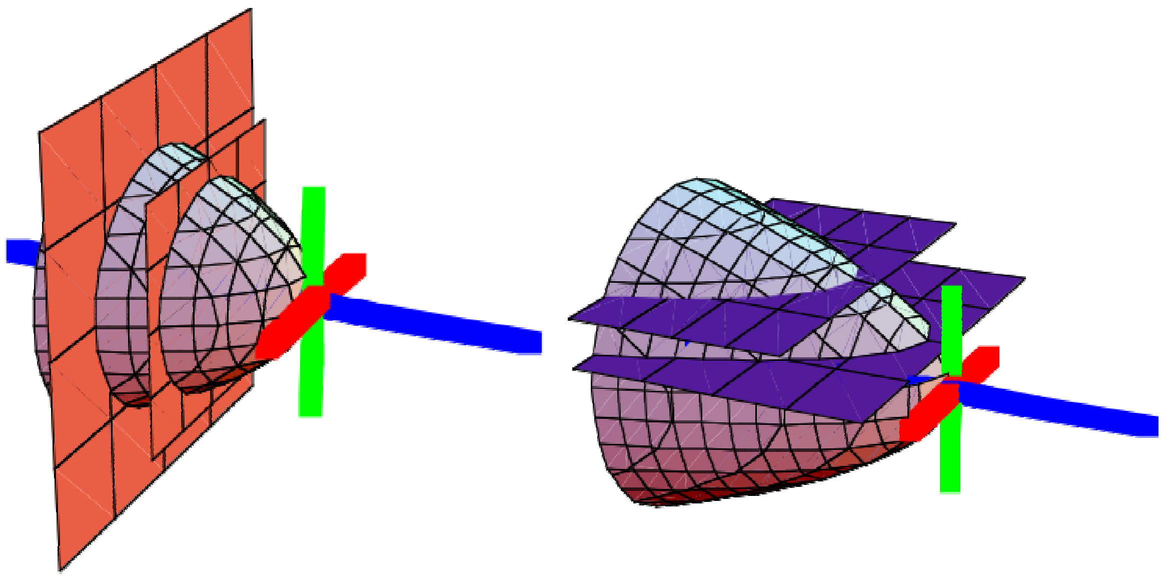
It follows from the above two observations that the vector $n_1 \times n_2$ is perpendicular to the plane we want. We compute it: $n_1 \times n_2 = \left(\begin{vmatrix} -5 & 2 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -5 \\ 4 & 1 \end{vmatrix} \right) = (3,9,21)$.

So, we have a point on the plane, namely (3,-1,5) and have a normal vector, namely (3,9,21). So, the equation of the plane is $(x-3, y+1, z-5) \bullet (3,9,21) = 0$, or $3(x-3)+9(y+1)+21(z-5)=0$.

2. [4 marks] Sketch the surface $x^2+z^2=-y$. Sketch at least two traces of that surface with planes of type $y=c$ (c various constants) and at least two traces with planes of type $z=c$ (c various constant).

Solution.

Traces of type $y=c$ yield circles (see the picture on the left). Traces of type $z=c$ yield parabolas (picture to the right). The surface is shown in both pictures.



3. [8 marks]

[1.5] (a) Find the rectangular coordinates of the point $(r,\theta,z)=\left(1,\frac{\pi}{4},1\right)$ given in cylindrical coordinates.

[1.5] (b) Find the rectangular coordinates of the point $(\rho,\theta,\phi)=\left(\frac{1}{2},\frac{3\pi}{2},\frac{\pi}{4}\right)$ given in spherical coordinates.

[2] (c) Find the cylindrical coordinates of the point $(x,y,z)=(3,-3,4)$ given in rectangular coordinates.

[2] (d) Find the spherical coordinates of the point $(x,y,z)=(0,6,8)$ given in rectangular coordinates. Do not simplify here (specifically, instead of finding ϕ, find, say, $\cos\phi$).

[1] Plot all of the above points in a single coordinate system.

Solution.

(a) $x=r\cos\theta=1\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$**,** $y=r\sin\theta=1\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$**,** z=1. So, the rectangular coordinates are $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1\right)$.

(b) $x = \rho \sin\phi \cos\theta = \frac{1}{2}\sin\frac{\pi}{4}\cos\frac{3\pi}{2} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\cdot 0 = 0$,

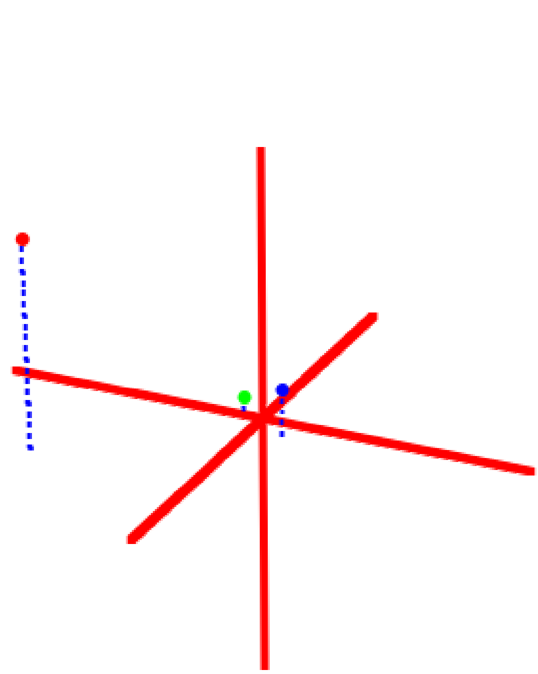
$y = \rho \sin\phi \sin\theta = \frac{1}{2}\sin\frac{\pi}{4}\sin\frac{3\pi}{2} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\cdot(-1) = -\frac{\sqrt{2}}{4}$,

$z = \rho\cos\phi = \frac{1}{2}\cos\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$. So the rectangular coordinates are $\left(0, -\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$

(c) The transition from rectangular to cylindrical coordinates is given by $x^2 + y^2 = r^2$, $\tan\theta = \frac{y}{x}$ and z=z. The first equation tells us that $r = \sqrt{3^2 + (-3)^2} = \sqrt{18}$. From $\tan\theta = \frac{y}{x}$ we get that $\tan\theta = \frac{-3}{3} = -1$ and so $\theta = \frac{7\pi}{4}$ (Note that $\theta = \frac{3\pi}{4}$ is also a solution of $\tan\theta = -1$. However, we do know that the point with coordinates x=3 and y=-3 is in the fourth quadrant of the xy-plane (which is where $\theta = \frac{7\pi}{4}$ leads us), while $\theta = \frac{3\pi}{4}$ would take us to the second quadrant.) Finally, obviously z=4. So, the cylindrical coordinates of the given point are $\left(\sqrt{18}, \frac{7\pi}{4}, 4\right)$.

(d) First of all we have that $\rho^2 = x^2 + y^2 + z^2$, from where we find that $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 6^2 + 8^2} = 10$. Secondly, since the given point is on the yz-plane (the first quadrant of it), we see that $\theta = \frac{\pi}{2}$. And finally, since $\cos\phi = \frac{z}{\rho}$, we compute $\cos\phi = \frac{8}{10}$.

Here is a plot of all of the four points: the first one (a) is in blue, the second (b) is green, the third (c) is red and the fourth one (d) is orange.



4. [8 marks].

[3] (a) Find all points on the curve $\vec{\mathbf{r}}(t)=\left(3t^6,4t^2-1,t^4-8t^2\right)$ where that curve has horizontal tangent lines.

[3] (b) Suppose l is any line perpendicular to the plane $x+y+z=2$. Find the angle between that line and the tangent vector to the curve $\vec{\mathbf{r}}(t)=\left(t^2-1,t^3+t,3+t^2\right)$ when $t=-1$.

[2] (c) Find the unit tangent vector of the curve $x=t$, $z=t^2$ at the point $(2,4)$.

Solution.

(a) Compute $\vec{\mathbf{r}}'(t)=\left(18t^5,8t,4t^3-16t\right)$. This vector is horizontal only if its third coordinate is 0, That gives us $4t^3-16t=0$ and solving this gives us three solutions: t=0, t=2 and t=-2. The corresponding points on the curve (which we get by substituting these values of t in $\vec{\mathbf{r}}(t)=\left(3t^6,4t^2-1,t^4-8t^2\right)$) are (0, -1, 0), ((3)(64), 15, -16) (Note that we reach the second point both when t=2 and when t=-2.)

(b) Since l is perpendicular to the given plane, it is parallel to the normal vector of that plane. So, l is parallel to the vector **u**=(1,1,1). Consequently we search for the angle between that vector and the tangent vector to the curve $\vec{\mathbf{r}}(t)=\left(t^2-1,t^3+t,3+t^2\right)$ when $t=-1$. One tangent vector to the curve $\vec{\mathbf{r}}(t)=\left(t^2-1,t^3+t,3+t^2\right)$ when $t=-1$ is $\vec{\mathbf{r}}'(t)$ when $t=-1$. We compute $\vec{\mathbf{r}}'(t)=\left(2t,3t^2+1,2t\right)$ and at $t=-1$ we get $\vec{\mathbf{r}}'(-1)=(-2,4,-2)$. Now, the angle θ between **u**=(1,1,1) and $\vec{\mathbf{r}}'(-1)=(-2,4,-2)$ can be found from the formula $\cos\theta=\dfrac{(1,1,1)\bullet(-2,4,-2)}{|(1,1,1)|\;|(-2,4,-2)|}=\dfrac{0}{\text{whatever but not 0}}=0$. So, $\theta=\dfrac{\pi}{2}$.

(c) This is the two dimensional curve $\vec{\mathbf{r}}(t)=\left(t,t^2\right)$. A tangent vector at any point is $\vec{\mathbf{r}}'(t)=\left(1,2t\right)$. The point (2,4) on the curve is attained when t=2. At that point we have $\vec{\mathbf{r}}'(t)=\left(1,2(2)\right)=(1,4)$. The unit vector is $T(t)=\dfrac{\vec{\mathbf{r}}'(2)}{|\vec{\mathbf{r}}'(2)|}=\dfrac{(1,4)}{\sqrt{1+16}}=\dfrac{1}{\sqrt{17}}(1,4)$.