Solutions of 136-170 problem worksheet n.9 ①

1 (a) $\int \sin 2x \sin 4x \, dx$

$= \frac{1}{2} \int [\cos(2x - 4x) - \cos(2x + 4x)] dx$

$= \frac{1}{2} \int [\cos(-2x) - \cos 6x] \, dx$

$= \frac{1}{2} \int [\cos 2x - \cos 6x] \, dx$

$= \frac{1}{2} \left[ \frac{\sin 2x}{2} - \frac{\sin 6x}{6} \right] + C$

(b) $\int \cos 7x \sin x \, dx = \frac{1}{2} \int [\sin(x - 7x) + \sin(x + 7x)] dx$

$= \frac{1}{2} \int [\sin(-6x) + \sin(8x)] \, dx$

$= \frac{1}{2} \int [-\sin 6x + \sin 8x] dx$

$= \frac{1}{2} \left[ \frac{\cos 6x}{6} - \frac{\cos 8x}{8} \right] + C$

(c) Let $u = x+1$. Then $du = dx$ &

$$\int \cos(x+1)\cos(3x+3)\,dx = \int \cos u \cos 3u \, du$$

$$= \frac{1}{2}\int [\cos(u-3u) + \cos(u+3u)\,du$$

$$= \frac{1}{2}\int (\cos 2u + \cos 4u)\,du$$

$$= \frac{1}{2}\left[-\frac{\sin 2u}{2} - \frac{\sin 4u}{4}\right] + C$$

$$= -\frac{1}{2}\left[\frac{\sin(2x+2)}{2} + \frac{\sin(4x+4)}{4}\right] + C$$

2 (a) Let $x = \sec\theta$, where $0 \leq \theta \leq \pi/2$ or $\pi \leq \theta < 3\pi/2$. Then $dx = \sec\theta \tan\theta \, d\theta$

and $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = |\tan\theta| = \tan\theta$

Since $\tan\theta$ is not negative on

$[0, \pi/2) \cup [\pi, 3\pi/2)$.

So $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan\theta}{\sec^4\theta}\sec\theta\tan\theta\,d\theta$

$\displaystyle = \int \sin^2\theta\cos\theta\,d\theta$

Let $u = \sin\theta$, Then $du = \cos\theta\,d\theta$. So

$\displaystyle = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sin^3\theta + C.$

Since $x = \sec\theta$, we have:

$\cos\theta = \frac{1}{x} \Rightarrow \sin\theta = \sqrt{1-\frac{1}{x^2}}$ if $0 \leq \theta < \pi/2$

and $\sin\theta = -\sqrt{1-\frac{1}{x^2}}$ if $\pi \leq \theta < \frac{3\pi}{2}$.

So if $x>0$, then $\sin\theta = \frac{\sqrt{x^2-1}}{|x|} = \frac{\sqrt{x^2-1}}{x}$.

if $x<0$, then $\sin\theta = -\frac{\sqrt{x^2-1}}{|x|} = -\frac{\sqrt{x^2-1}}{-x} = \frac{\sqrt{x^2-1}}{x}$.

Hence, in any case, the integral is

$\displaystyle \tfrac{1}{3}\cdot\frac{(x^2-1)^{3/2}}{x^3} + C$

(b) Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta \, d\theta$.

$$\int_0^2 x^3\sqrt{x^2+4} = \int_0^{\pi/4} 8\tan^3\theta \cdot 2\sec\theta \cdot 2\sec^2\theta \, d\theta$$

$$= 32\int_0^{\pi/4} \tan^2\theta \sec^2\theta \tan\theta\sec\theta \, d\theta$$

$$= 32\int_0^{\pi/4} (\sec^2\theta - 1)\sec^2\theta \cdot \tan\theta\sec\theta \, d\theta$$

Let $u = \sec\theta$. Then $du = \tan\theta\sec\theta \, d\theta$.

$$= 32\int_1^{\sqrt{2}} (u^2-1)u^2 \, du = 32\int_1^{\sqrt{2}} [u^4 - u^2] du$$

$$= 32\left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^{\sqrt{2}} = 32\left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3}\right.$$

$$\left. - \frac{1}{5} + \frac{1}{3}\right) = 32\left(\frac{2\sqrt{2}+2}{15}\right)$$

$$= \frac{64}{15}(\sqrt{2}+1)$$

(c) Let $x = \sqrt{5}\sin\theta$ ~~then~~, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Then $dx = \sqrt{5}\cos\theta\, d\theta$. So

$$\int \frac{dx}{x^2\sqrt{5-x^2}} = \int \frac{\cancel{\sqrt{5}}\cos\theta\, d\theta}{5\sin^2\theta \cdot \cancel{\sqrt{5}}\cos\theta} =$$

$$\frac{1}{5}\int \csc^2\theta\, d\theta = -\frac{1}{5}\cot\theta + C$$

$$= -\frac{1}{5}\cdot\frac{\sqrt{5-x^2}}{x} + C$$

(d) Let $u = e^x$. Then $du = e^x dx$. So $dx = \frac{du}{u}$ &

$$\int \frac{e^{-x}}{\sqrt{1+e^{2x}}}dx = \int \frac{1}{u\sqrt{1+u^2}} \cdot \frac{du}{u} = \int \frac{du}{u^2\sqrt{1+u^2}}$$

For the rest, see Example 3 / page 485 in the book.

(e) Let $u = x-1$. Then $du = dx$ &

$$\int \sqrt{4-(x-1)^2}\,dx = \int \sqrt{4-u^2}\,du.$$

Let $u = 2\sin\theta$, where $-\frac{\pi}{2} < \theta < \pi/2$.

Then $du = 2\cos\theta \, d\theta$ and

$$\int \sqrt{4-u^2}\, du = \int 2\cos\theta \cdot 2\cos\theta \, d\theta$$

$$= 4\int \cos^2\theta \, d\theta = 4\int \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= 2\left(\theta + \frac{\sin 2\theta}{2}\right) + C = 2\theta + \sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{u}{2}\right) + \frac{u}{2} \cdot \sqrt{1 - \frac{u^2}{4}} + C$$

$$= 2\sin^{-1}\left(\frac{x-1}{2}\right) + \frac{x-1}{2} \cdot \sqrt{1 - \frac{(x-1)^2}{4}} + C$$

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3 (a) $$\frac{x^3 + x^2 - 12x + 1}{x^2 + x - 12} = x + \frac{1}{x^2 + x - 12}$$

$$= x + \frac{1}{(x-3)(x-4)} = x + \frac{1}{7}\left(\frac{1}{x-3} - \frac{1}{x+4}\right).$$

So $$\int \cdots = \left[\frac{1}{2}x^2 + \frac{1}{7}\ln|x-3| - \frac{1}{7}\ln|x+4|\right]_0^2$$

$$= 2 + \frac{1}{7}\ln\frac{2}{9}$$

$$1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2.$$

$x = 0 \Rightarrow \boxed{B = 1}$. $x = 1 \Rightarrow \boxed{D = 1}$.

So $Ax(x-1)^2 + Cx^2(x-1) = 1 - (x-1)^2 - x^2 \Rightarrow$

$\cancel{x(x-1)}[A(x-1) + Cx] = -2x^2 + 2x = 2\cancel{x(x-1)} \Rightarrow$

$A(x-1) + Cx = -2$ if $x \neq 0, 1$

So $A + C = 0$ & $-A = -2 \Rightarrow \cancel{A = C = X}$

$\Rightarrow \boxed{A = 2}$ & $\boxed{C = -2}$. So

$$\int \frac{dx}{x^2(x-1)^2} = \int \left[\frac{2}{x} + \frac{1}{x^2} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right] dx$$

$$= 2\ln|x| - \frac{1}{x} - 2\ln|x-1| - \frac{1}{x-1} + C$$