Solutions to Tutorial 8

(1) b) Let $u = \ln x \quad du = \frac{1}{x}dx$

$dv = x^3 \quad \cancel{dv} = \frac{x^4}{4}$

$\therefore \; I = \frac{x^4 \ln x}{4} - \int \frac{x^4}{4}\left(\frac{1}{x}\right)dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$

$= \frac{1}{4}x^4 \ln x - \frac{x^4}{16} + C$

c) Let $u = \sin 3x \quad du = 3\cos 3x\,dx$

$dv = e^{2x}dx \quad v = \frac{1}{2}e^{2x}$

$I = \sin 3x\left(\frac{1}{2}e^{2x}\right) - \int 3\cos 3x\left(\frac{1}{2}\right)e^{2x}dx$

$= \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{2}\int \cos 3x\, e^{2x}\,dx$

Let $u = \cos 3x \quad du = -3\sin 3x\,dx$

$dv = e^{2x}dx \quad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{2}\left[\frac{1}{2}\cos 3x\, e^{2x} - \int(-3\sin 3x\, e^{2x}\,dx\right.$

$= \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{4}\cos 3x\, e^{2x} + \frac{3}{2}I$

$\Rightarrow \frac{5}{2}I = \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{4}\cos 3x\, e^{2x} +$

$\Rightarrow I = \frac{1}{5}\sin 3x\, e^{2x} - \frac{3}{10}\cos 3x\, e^{2x} + C$

d) $\int_0^{\pi/2} x \sin^2 x \, dx$

Let $u = x$ $\quad du = dx$

$dv = \sin^2 x \, dx \quad v = \int \sin^2 x \, dx$

$= \int \frac{1 - \cos 2x}{2}$

$= \frac{x}{2} - \frac{\sin 2x}{4}$

$$I = \left(x\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]\right)\Bigg]_0^{\pi/2} - \int_0^{\pi/2}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)dx$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2} - \frac{\sin \pi}{4}\right] - \left(0\left(\frac{0}{2} - \frac{\sin 0}{2}\right)\right) - \frac{x^2}{4} + \frac{\cos 2x}{8}\Bigg]_0^{\pi/2}$$

$$= \frac{\pi^2}{4} - \left(\frac{\left(\frac{\pi}{2}\right)^2}{4} + \frac{\cos \pi}{8}\right)$$

$$= \frac{\pi^2}{4} - \frac{\pi^2}{16} + \left(\frac{-1}{8}\right) = \frac{3\pi^2}{16} - \frac{1}{8}$$

e) Let $u = \sec^{-1} x \quad du = \frac{1}{x\sqrt{x^2-1}}\,dx$

$dv = x\,dx \quad v = x^2/2$

$$I = \frac{x^2}{2}\sec^{-1}x\Bigg]_1^2 - \int_1^2 \frac{x^2}{2}\frac{1}{x\sqrt{x^2-1}}\,dx$$

$$= \frac{2^2}{2}\sec^{-1}(2) - \frac{1^2}{2}\sec^{-1}(1) - \frac{1}{2}\int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx$$

et $u = x^2 - 1$

$du = 2x\,dx$

$x = 1 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 3$

$$= 2\left(\frac{\pi}{3}\right) - \frac{1}{2}(0) - \frac{1}{2}\int_0^3 \frac{1}{2\sqrt{u}}\,du$$

$$= \frac{2\pi}{3} - \frac{1}{2}\sqrt{u}\Bigg]_0^3 = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

① f) First let $z = \sqrt{x}$ $dz = \frac{1}{2\sqrt{x}} dx$ $x = z^2$

$x=0 \Rightarrow z=0$
$x=4 \Rightarrow z=2$

So $I = \int_0^2 \sqrt{x} e^{\sqrt{x}} (2\sqrt{x}) dz$

$= 2\int_0^2 x e^{\sqrt{x}} dz$

$= 2\int_0^2 z^2 e^{z} dz$ ← Could have used $x = z^2$ $dx = 2z\,dz$

let $u = z^2$ $du = 2z\,dz$
$dv = e^z$ $v = e^z$

$= 2\left(z^2 e^z\Big]_0^2 - 2\int_0^2 z e^z dz\right)$

$= 2(2^2 e^2 - 0^2 e^0) - 4\int_0^2 z e^z dz$ let $u = z$ $du = dz$
$dv = e^z$ $v = e^z$

$= 8e^2 - 4\left(\left[z e^z\right]_0^2 - \int_0^2 e^z dz\right)$

$= 8e^2 - 4(2e^2 - 0e^0) + 4\int_0^2 e^z dz$

$= 8e^2 - 8e^2 + 4\, e^z\Big]_0^2$

$= 4(e^2 - e^0) = 4(e^2 - 1)$

② let $u = (\ln x)^n$ $du = n(\ln x)^{n-1}\left(\frac{1}{x}\right)dx$
$dv = dx$ $v = x$

So $\int (\ln x)^n dx = x(\ln x)^n - \int x\left(n(\ln x)^{n-1}\left(\frac{1}{x}\right)\right)dx$

$= x(\ln x)^n - n\int(\ln x)^{n-1} dx$

So

$\int (\ln x)^4 \, dx$

$= x(\ln x)^4 - 4\int(\ln x)^3 \, dx$

$= x(\ln x)^4 - 4\left[x(\ln x)^3 - 3\int(\ln x)^2 dx\right]$

$= x(\ln x)^4 - 4x(\ln x)^3 + 12\left[x(\ln x)^2 - 2\int(\ln x) dx\right]$

$= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24\left(x\ln x - \int 1 dx\right)$

$= x(\ln x)^4 - 4x(\ln x)^3 + 12x\ln x - 24\ln x + 24x + C$

3 a) let $u = \sin x$

$du = \cos x$

So $\int \sin^3 x \cos^3 x \, dx = \int \sin^3 x \cos^2 x (\cos x \, dx)$

$= \int \sin^3 x (1 - \sin^2 x) \cos x \, dx)$

$= \int u^3(1-u^2) \, du$

$= \int (u^3 - u^5) du$

$= \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$

b) $\int \tan^2 x \sec^2 x \, dx$

let $u = \tan x$

$du = \sec^2 x \, dx$

$= \int u^2 du = \frac{u^3}{3} + C$

$= \frac{\tan^3 x}{3} + C$

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(3) c) $\int \cot^3 x \csc x$

let $u = \csc x \, du$
$du = -\csc x \cot x$

$= \int \cot^2 x (\csc x \cot x \, dx)$

$= \int (\csc^2 x - 1)(\csc x \cot x \, dx)$

$= - \int (u^2 - 1) du$

$= -\frac{u^3}{3} + u + C$

$= \frac{-\csc^3 x}{3} + \csc x + C$

d) $\int \sin^3 x \, dx$

let $u = \cos x$
$du = \sin x \, dx$

$= \int \sin^2 x \, (\sin x \, dx)$

$= \int (1 - \cos^2 x)(\sin x \, dx)$

$= \int (1 - u^2)(-du)$

$= \int (u^2 - 1) du$

$= \frac{u^3}{3} - u + C$

$= \frac{\cos^3 x}{3} - \cos x + C$

c) $\int \tan^2 x \sec x$ (can't split a $\sec x \tan x$ or $\sec^2 x$)

let $u = \tan x$ $\quad dv = \sec x \tan x \, dx$
$du = \sec^2 x \, dx$ $\quad v = \sec x$

$I = \tan x \sec x - \int \sec^3 x \, dx$
$= \tan x \sec x - \int \sec x (\tan^2 x + 1) \, dx$
$= \tan x \sec x - \int \sec x \tan^2 x \, dx - \int \sec x \, dx$
$2I = \tan x \sec x - \ln|\sec x + \tan x| + C$

(Note: $\int \sec x = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x}$

let $u = \sec x + \tan x$
$du = \sec x \tan x + \sec^2 x$

$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$)