Mathematics 170

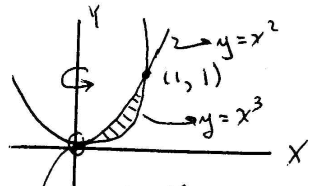
Tutorial Worksheet No. 7

Wed. March 2 to Tues March 8, 2005

SOLUTIONS

1. Calculate the volume $V$ of the solid obtained by rotating about the Y-axis the region $R$ of the first quadrant bounded by $y = x^2$ and $y = x^3$.

By cylindrical shells: The curves intersect at points whose $x$-coordinates satisfy the equation $x^2 = x^3$, i.e. $x^3 - x^2 = 0$, i.e. $x^2(x-1) = 0$. This happens at $x = 0$ and $x = 1$.



R is the shaded region

Observe that the parabola lies above the cubic curve if $0 \le x \le 1$. Thus, using the basic formula,

$$V = 2\pi \int_0^1 x(x^2 - x^3)\,dx$$

$$= 2\pi \int_0^1 (x^3 - x^4)\,dx = 2\pi \left( \frac{x^4}{4} - \frac{x^5}{5} \Big|_0^1 \right)$$

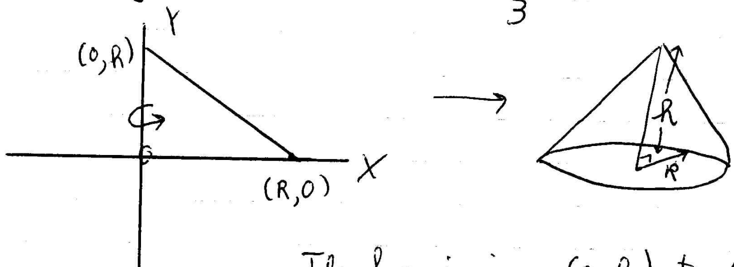
$$= 2\pi \left[ \left( \frac{1}{4} - \frac{1}{5} \right) - (0 - 0) \right] = 2\pi \left( \frac{1}{20} \right) = \frac{\pi}{10} \text{ cubic units}$$

By "washers": We integrate re $y$. The curves have equations $x = y^{\frac{1}{3}}$ and $x = y^{\frac{1}{2}}$, and $y^{\frac{1}{3}} \ge y^{\frac{1}{2}}$ if $0 \le y \le 1$. The values of $y$ range from 0 to 1. Thus, using the basic formula,

$$V = \int_0^1 \pi \left[ (y^{\frac{1}{3}})^2 - (y^{\frac{1}{2}})^2 \right] dy = \pi \int_0^1 (y^{\frac{2}{3}} - y)\,dy = \pi \left( \frac{y^{\frac{5}{3}}}{5/3} - \frac{y^2}{2} \Big|_0^1 \right)$$

$$= \pi \left( \left( \frac{3}{5} - \frac{1}{2} \right) - (0 - 0) \right) = \frac{\pi}{10} \text{ cubic units.}$$

2. By rotating the triangle with vertices as shown about the Y-axis, show that a right circular cone with base radius $R$ and height $h$ has volume $\frac{1}{3}\pi R^2 h$.




The line joining $(0,h)$ to $(R,0)$ has equation

$$\frac{y-h}{x-0} = \frac{h-0}{0-R}, \text{ ie } \quad y-h = -\frac{h}{R}x$$

$$\text{i.e } \quad y = -\frac{h}{R}x + h$$

$\therefore f(x) = -\frac{h}{R}x + h$. Using the basic formula (integrating re $x$ and using cylindrical shells)

we get $\quad V = \int_0^R 2\pi x \left( \left( -\frac{hx}{R} + h \right) - 0 \right) dx$

$$= \frac{-2\pi h}{R} \int_0^R x^2\,dx + 2\pi h \int_0^R x\,dx$$

$$= \left. \frac{-2\pi h x^3}{3R} \right|_0^R + \left. \frac{2\pi h x^2}{2} \right|_0^R$$

$$= \frac{-2\pi h R^2}{3} - 0 + \pi h R^2 - 0 = \frac{1}{3}\pi h R^2 \text{ cubic units.}$$

3. (a) $\dfrac{d}{dx}\left(\sin^{-1}(\sqrt{x})\right) = \dfrac{d}{dx}\left(\sin^{-1}(x^{\frac{1}{2}})\right)$

$$= \frac{1}{\sqrt{1-(x^{\frac{1}{2}})^2}} \cdot \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \frac{1}{2\sqrt{1-x}\;\sqrt{x}}$$

When $x = 0.64$, $\sqrt{x} = 0.8$ and $\sqrt{1-x} = \sqrt{1-0.64} = \sqrt{0.36} = 0.6$

and so $\dfrac{d}{dx}\left(\sin^{-1}(\sqrt{x})\right)\Big|_{x=0.64} = \dfrac{1}{2\,(0.6)(0.8)} = \dfrac{1}{0.96}$

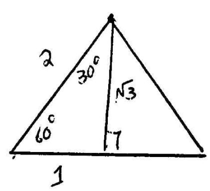
$= \dfrac{100}{96} = \dfrac{25}{24}$.

(b) $\dfrac{d}{dx}\left(\tan^{-1}(x)\right)^3 = 3\left(\tan^{-1}(x)\right)^2 \cdot \dfrac{d}{dx}\left(\tan^{-1}(x)\right)$

$$= \frac{3\left(\tan^{-1}(x)\right)^2}{1+x^2}$$

When $x = \sqrt{3}$

$\tan^{-1}x = \tan^{-1}\sqrt{3} = 60^\circ = \dfrac{\pi}{3}$ and $1+x^2 = 4$.




$\therefore \dfrac{d}{dx}\left(\tan^{-1}(x)\right)^3\Big|_{x=\sqrt{3}} = \dfrac{3\left(\frac{\pi}{3}\right)^2}{4}$

$= \dfrac{3\pi^2}{4\times 9} = \dfrac{\pi^2}{12}$

3. (c) $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9x^2}\,dx$.

If we let $u = 3x$, then $\dfrac{1}{1+9x^2} = \dfrac{1}{1+u^2}$, and $\displaystyle\int \frac{1}{1+u^2}\,du = \tan^{-1}(u)$.

If $u = 3x$ then $\dfrac{du}{dx} = 3$, i.e. $du = 3\,dx$, $\therefore \frac{1}{3}du = dx$

When $x = 0$, $u = 0$ and when $x = \frac{1}{\sqrt{3}}$; then $u = \frac{3}{\sqrt{3}} = \sqrt{3}$.

Thus $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9x^2}\,dx = \int_0^{\sqrt{3}} \frac{1}{1+u^2}\cdot\frac{1}{3}\,du$

$$= \frac{1}{3}\int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du = \frac{1}{3}\tan^{-1}(u)\Big|_0^{\sqrt{3}} = \frac{1}{3}\tan^{-1}(\sqrt{3}) - \frac{1}{3}\tan^{-1}(0)$$

$$= \frac{1}{3}\left(\frac{\pi}{3}\right) - 0 = \frac{\pi}{9}$$

see 3(b)

4. $\tan^{-1}(-1) = -\frac{\pi}{4}$ so $\cos\left(\tan^{-1}(-1)\right) = \cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

so $\sin^{-1}\left(\cos\left(\tan^{-1}(-1)\right)\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$, while

$\sin^{-1}(-1) = -\frac{\pi}{2}$ so $\cos\left(\sin^{-1}(-1)\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

so $\tan^{-1}\left(\cos\left(\sin^{-1}(-1)\right)\right) = \tan^{-1}(0) = 0$.

$\therefore \sin^{-1}\left(\cos\left(\tan^{-1}(-1)\right)\right) - \tan^{-1}\left(\cos\left(\sin^{-1}(-1)\right)\right) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

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