Problem Worksheet 6 Feb. 20 – 24, 2006

Solutions

1. (a) Find the area of the region of the plane bounded by the parametric curves and the x-axis

$$x = \frac{t^2+1}{2}, \quad y = e^{t^2} \quad (1 \le t \le 3).$$

We see that as t increases from 1 to 3, x increases from 1 to 5 and y increases from e to e^9

Using the basic formula (page 662 of text)

$$A = \int_{t=1}^{t=3} (\overset{g(t)}{e^{t^2}})(\overset{f'(t)}{t})\,dt \qquad = \frac{1}{2}\int_1^3 (e^{t^2})(2t\,dt)$$

[Here $x = f(t) = \frac{t^2+1}{2}$ and $y = g(t) = e^{t^2}$

so $g(t)f'(t) = (e^{t^2})(t)$

[Let $u(t) = t^2$

$\therefore \frac{du}{dt} = 2t$

so $du = 2t\,dt$]

When $t=1$, $u = 1^2 = 1$. When $t=3$, $u = 3^2 = 9$

$$= \frac{1}{2}\int_1^9 e^u\,du = \frac{1}{2} e^u \Big|_1^9 = \frac{e^9 - e}{2} \text{ sq. units.}$$

1 (b). Assume $a > 0$ and $b > 0$

The answer is $\frac{ab}{2}$ [When $\theta = \pi$, $x = -a$; when $\theta = 0$, $x = a$. So, as θ decreases from π to 0, x increases from $-a$ to a]

$x = f(\theta) = a\cos\theta$

$y = g(\theta) = b\sin\theta$

$$A = \int_{\pi}^{0} \underbrace{(-a\sin\theta)}_{f'(\theta)} \underbrace{(b\sin\theta)}_{g(\theta)}\, d\theta = -ab\int_{\pi}^{0} \sin^2\theta\, d\theta$$

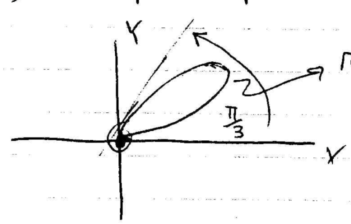
$$= ab\int_0^{\pi} \sin^2\theta\, d\theta = ab\int_0^{\pi} \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= ab\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\Big|_0^{\pi}\right] = ab\left[\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) - \left(\frac{0}{2} - \frac{\sin 0}{4}\right)\right]$$

$$= ab\left[\left(\frac{\pi}{2} - 0\right) - (0-0)\right] = \frac{\pi ab}{2}.$$

2. When $\theta = 0$, $r = 0$. When $\theta = \frac{\pi}{3}$, $r = 4\sin\pi = 0$

So, one loop corresponds to values of θ between 0 and $\frac{\pi}{3}$



$r = 4\sin(3\theta)$

$$\therefore A = \frac{1}{2}\int_0^{\frac{\pi}{3}} [4\sin(3\theta)]^2\, d\theta$$

$$= 8\int_0^{\frac{\pi}{3}} \sin^2(3\theta)\, d\theta$$

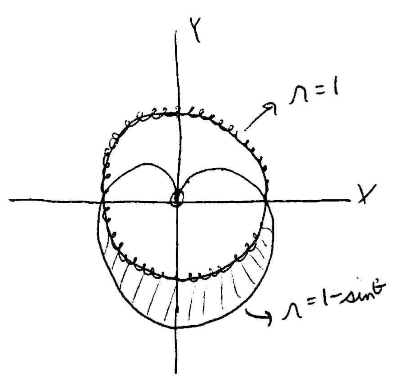
$$= 8\int_0^{\frac{\pi}{3}} \frac{1-\cos(4\theta)}{2}\, d\theta = 8\left[\frac{\theta}{2} - \frac{1}{8}\sin(4\theta)\Big|_0^{\frac{\pi}{3}}\right]$$

$$= 8\left[\left(\frac{\pi}{6} - \frac{1}{8}\sin\left(\frac{4\pi}{3}\right)\right) - (0-0)\right]$$

$$= 8\left(\frac{\pi}{6} - \left(\frac{1}{8}\right)\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{4\pi}{3} + \frac{\sqrt{3}}{2} \text{ sq. units}$$

3 (a) Find the area of the region outside $r=1$ and inside $r = 1-\sin\theta$.



$r=1$ is a circle centred at the origin and with radius 1. This curve meets $r = 1-\sin\theta$ when $1-\sin\theta = 1$, i.e. $\sin\theta = 0$, i.e. $\theta = 0$ and $\theta = \pi$. If $0 \le \theta \le \pi$ then $\sin\theta \ge 0$ and $0 \le 1-\sin\theta \le 1$. If $\pi \le \theta \le 2\pi$, then $\sin\theta \le 0$ and $1 \le 1-\sin\theta \le 2$

$\therefore$ $r = 1-\sin\theta$ lies outside $r=1$ for $\pi \le \theta \le 2\pi$.

$\therefore$ the area of the region (cross-hatched in the diagram) is

$$\frac{1}{2}\int_{\pi}^{2\pi} (1-\sin\theta)^2 \, d\theta - \underbrace{\frac{1}{2}\int_{\pi}^{2\pi} 1^2 \, d\theta}_{\text{area of a semi-circle of radius 1}}$$

$$= \frac{1}{2}\int_{\pi}^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\, d\theta - \frac{\pi}{2}.$$

Now $\displaystyle\int_\pi^{2\pi}(1-2\sin\theta)\,d\theta = \theta+2\cos\theta\Big|_\pi^{2\pi} = (2\pi+2(\cos 2\pi)) - (\pi+2\cos\pi)$

$= (2\pi+2)-(\pi-2) = \pi+4$

and $\displaystyle\int_\pi^{2\pi}\sin^2\theta\,d\theta = \int_\pi^{2\pi}\frac{1-\cos(2\theta)}{2}\,d\theta = \frac{\theta}{2}-\frac{\sin(2\theta)}{4}\Big|_\pi^{2\pi}$

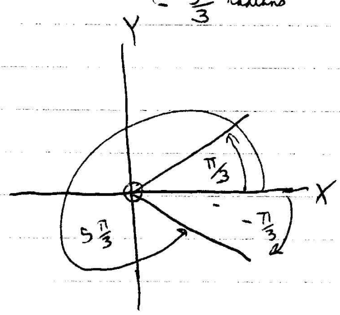
$= \left(\pi - \dfrac{\sin(4\pi)}{4}\right) - \left(\dfrac{\pi}{2}-\dfrac{\sin(2\pi)}{4}\right) = \dfrac{\pi}{2}$.

$\therefore$ the area is $\pi+4+\dfrac{\pi}{2} = \dfrac{3\pi}{2}+4 = \dfrac{3\pi+8}{2}$ sq. units.

(b) The curve $r=1+\cos\theta$ meets the curve $r=3\cos\theta$ when $1+\cos\theta=3\cos\theta$, i.e. $1=2\cos\theta$, i.e. $\cos\theta=\frac{1}{2}$, i.e. $\theta=-\frac{\pi}{3}$ and $\theta=\frac{\pi}{3}$

$= \frac{5\pi}{3}$ radians

When $\theta=0$, $3\cos\theta=3$ and $1+\cos\theta=2$ so $r=3\cos\theta$ is inside $r=1+\cos\theta$ for $\frac{\pi}{3}\le\theta\le\frac{5\pi}{3}$



$\therefore A = \dfrac{1}{2}\displaystyle\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}(1+\cos\theta)^2\,d\theta - \dfrac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}(3\cos\theta)^2\,d\theta$

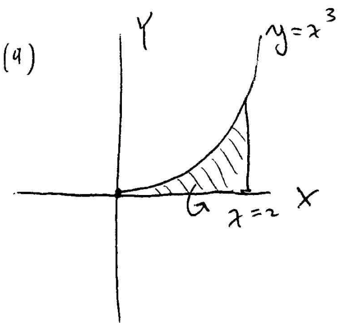
$= \dfrac{1}{2}\displaystyle\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}(1+2\cos\theta-8\cos^2\theta)\,d\theta = \dfrac{1}{2}\left[\theta+2\sin\theta-8\left(\dfrac{\sin(2\theta)}{4}-\dfrac{\theta}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$

[Note: $\int \cos^2\theta \, d\theta = \int \frac{\cos(2\theta)-1}{2} \, d\theta = \frac{1}{4}\sin(2\theta) - \frac{\theta}{2}$]

$$= \frac{1}{2}\left[\frac{5\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) - 8\left(\frac{\sin\left(\frac{10\pi}{3}\right)}{4} - \frac{5\pi}{6} \right) \right]$$

$$- \frac{1}{2}\left[\frac{\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{\sin\left(\frac{2\pi}{3}\right)}{4} - \frac{\pi}{6} \right) \right] \quad = \text{etc.}$$

4. (a)



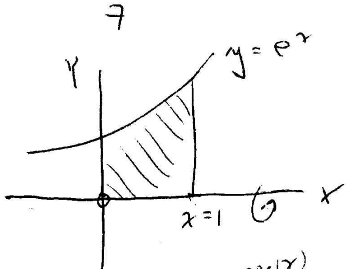
$y = x^3 \quad x = 2 \quad y = 0$

$$V = \int_0^2 \pi (x^3)^2 dx$$

$$= \int_0^2 \pi x^6 dx = \frac{\pi x^7}{7} \Big|_0^2$$

$$= \frac{128\pi}{7} \text{ units}.$$

(b)

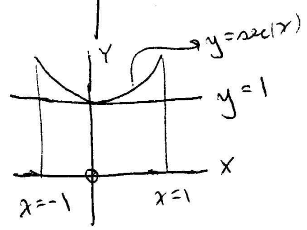


$$V = \int_0^1 \pi \, (e^x)^2 dx$$

$$= \pi \int_0^1 e^{2x} dx = \frac{\pi}{2} e^{2x} \Big|_0^1$$

$$= \frac{\pi e^2}{2} \text{ cubic units}$$

(c)



Note If $-1 \le x \le 1$, then $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ so x is in the 1st or 4th quadrant so $\sec x \ge 0$ so $\sec x = \frac{1}{\cos x} \ge 1$.

$V =$ (volume generated by rotating about the x-axis the region under $y = \sec x$ above x-axis, and between $x = -1$ and $x = 1$)

$-$ (volume generated by rotating about the x-axis the region between $y = 0$, $y = 1$, $x = -1$, and $x = 1$)

$$= \int_{-1}^{1} \pi (\sec(x))^2 \, dx - \int_{-1}^{1} \pi (1)^2 \, dx$$

$$= \pi \left[\int_{-1}^{1} \sec^2(x) \, dx - \int_{-1}^{1} 1 \, dx \right] = \pi \left[\tan(x) \Big|_{-1}^{1} - x \Big|_{-1}^{1} \right]$$

$$= \pi \left[[\tan(1) - \tan(-1)] - (1 - (-1)) \right]$$

$$= \pi \left[2\tan(1) - 2 \right]$$

[Remark: $\tan\left(\frac{\pi}{4}\right) = 1$ and $\frac{\pi}{4} < 1 < \frac{\pi}{2}$ so $\tan(1) > 1$

so $2\tan(1) - 2 > 0$.]

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