Mathematics 170

Worksheet No. 5 Solutions

1. (a) $\int x^3\sqrt{x^4+1}\,dx$.

Let $u = x^4+1$ $\therefore \frac{du}{dx} = 4x^3$ so $du = 4x^3\,dx$

$= \frac{1}{4}\int \sqrt{x^4+1}\,(4x^3dx)$

$= \frac{1}{4}\int u^{\frac{1}{2}}\,du = \frac{1}{4}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{6}(x^4+1)^{\frac{3}{2}} + C$

[Remark: If we had written $\frac{1}{4}\left(\frac{u^{\frac{3}{2}}}{3/2} + C\right) = \frac{1}{6}(x^4+1)^{\frac{3}{2}} + 4C$, this is the same as the first answer because C is an arbitrary constant if and only if $4C$ is an arbitrary constant.

(b) $\int \frac{\cos(x)}{\sin^5(x)}\,dx$.

Let $u = \sin(x)$ $\therefore \frac{du}{dx} = \cos(x)$ and $du = \cos(x)\,dx$

$= \int u^{-5}\,du = \frac{u^{-4}}{-4} + C = -\frac{1}{4\sin^4(x)} + C$

(c) $\int \frac{\ln(\ln(3x+2))}{(3x+2)\ln(3x+2)}\,dx$

Let $u = \ln(\ln(3x+2))$

$\therefore \frac{du}{dx} = \frac{1}{\ln(3x+2)}\cdot\frac{d}{dx}[\ln(3x+2)]$

$= \frac{1}{\ln(3x+2)}\cdot\frac{1}{3x+2}\cdot 3$

so $du = \frac{3}{(3x+2)(\ln(3x+2))}\,dx$

$= \frac{1}{3}\int \ln(\ln(3x+2))\cdot\frac{3\,dx}{(3x+2)\ln(3x+2)}$

$= \frac{1}{3}\int u\,du = \frac{u^2}{6} + C = \frac{[\ln(\ln(3x+2))]^2}{6} + C$

d) $\int e^{-\tan(5x)}\sec^2(5x)\,dx$.

Let $u = -\tan(5x)$ $\therefore \frac{du}{dx} = -5\sec^2(5x)$, $du = -5\sec^2(5x)\,dx$

$= -\frac{1}{5}\int e^u\,du = -\frac{e^u}{5} + C = -\frac{e^{-\tan(5x)}}{5} + C.$

2. (a) $\displaystyle\int_0^1 \frac{x}{\sqrt{3x^2+1}}\,dx$. Let $u = 3x^2+1$. When

When $x=1$, $u = 3(1)^2+1 =$

$\dfrac{du}{dx} = 6x$ so $du = 6x\,dx$

$$= \frac{1}{6}\int_0^1 \frac{6x\,dx}{\sqrt{3x^2+1}} = \frac{1}{6}\int_0^2 u^{-\frac{1}{2}}\,du = \frac{1}{6}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_0^2\right) = \frac{1}{3}\sqrt{u}\,\Bigg|_1^4 =$$

$$\frac{1}{3}\sqrt{4} - \frac{1}{3}\sqrt{1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

(b) $\displaystyle\int_{[\ln(2)]^2}^{[\ln(3)]^2} \frac{e^{\sqrt{x}}}{\sqrt{2x}}\,dx$. Let $u = \sqrt{x} = x^{\frac{1}{2}}$

When $x = (\ln(2))^2$, $u = \ln(2)$

When $x = [\ln(3)]^2$, $u = \ln(3)$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2}\sqrt{2x}}$$

so $du = \dfrac{1}{\sqrt{2}} \cdot \dfrac{dx}{\sqrt{2x}}$

So, our integral equals $\displaystyle\sqrt{2}\int_{[\ln(2)]^2}^{[\ln(3)]^2} e^{\sqrt{x}} \cdot \frac{1}{\sqrt{2}\sqrt{2x}}\,dx$

$$= \sqrt{2}\int_{\ln(2)}^{\ln(3)} e^u\,du = \sqrt{2}\,e^u\Big|_{\ln(2)}^{\ln(3)} = \sqrt{2}\,e^{\ln(3)} - \sqrt{2}\,e^{\ln(2)}$$

$$= \sqrt{2}(3) - \sqrt{2}(2) = \sqrt{2}.$$

(c) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos(2x)}{2+\sin(2x)}\,dx$. Let $v = 2+\sin(2x)$

When $x=0$, $v = 2+\sin(0) = 2$

When $x = \frac{\pi}{4}$, $v = 2+\sin\left(\frac{\pi}{2}\right) = 3$

$\dfrac{dv}{dx} = 2\cos(2x)$ so $dv = 2\cos(2x)\,dx$. Our integral is

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{2+\sin(2x)} \cdot 2\cos(2x)\,dx = \frac{1}{2}\int_2^3 \frac{1}{v}\,dv = \frac{1}{2}\ln|v|\Big|_2^3 = \frac{1}{2}(\ln 3 - \ln(2))$$

$$= \frac{1}{2}\ln\left(\frac{3}{2}\right) = \ln\sqrt{1.5}.$$

3. Let k be a positive constant. We know that the area of the region bounded by $y = \frac{1}{\sqrt{kx+1}}$, $x=0$, $x=2$, and the x-axis is 1.

$\therefore \int_0^2 \frac{dx}{\sqrt{kx+1}} = 1.$

Let $u = kx+1$ $\therefore \frac{du}{dx} = k$
so $du = k\,dx$
When $x=0$, $u=1$. When $x=2$, $u=2k+1$

So our integral is

$\frac{1}{k}\int_0^2 \frac{k\,dx}{\sqrt{kx+1}}$ (We can divide by k because we are told that $k>0$.)

$= \frac{1}{k}\int_1^{2k+1} u^{-\frac{1}{2}}\,du = \frac{1}{k}\left(2u^{\frac{1}{2}}\right)\Big|_1^{2k+1} = \frac{2}{k}\sqrt{2k+1} - \frac{2}{k}(1)$

So we have: $\frac{2}{k}\sqrt{2k+1} - \frac{2}{k} = 1$

Thus $2\sqrt{2k+1} = k+2$. Square and get $4(2k+1) = k^2+4k+4$

$\therefore$ $k^2 - 4k = 0$, or $k(k-4)=0$, which has solutions $k=0$ and $k=4$. As $k>0$, $k=4$ is the correct solution.

4. Let $a>0$. Note that $\int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3+1}\,dx = \frac{1}{3}\int \frac{1}{u}\,du$ where we let $u = x^3+1$

$= \frac{1}{3}\ln|u| = \frac{1}{3}\ln|x^3+1|$.

$\therefore$ if $a>0$, we have $2\int_0^a \frac{x^2}{x^3+1}\,dx = \int_0^{2a} \frac{x^2}{x^3+1}\,dx$ if and only if

$2\left[\frac{1}{3}\ln(x^3+1)\Big|_0^a\right] = \frac{1}{3}\ln(x^3+1)\Big|_0^{2a}$ if and only if $\ln(a^3+1) - \ln 1 = \frac{1}{2}\left[\ln(8a^3+1) - \ln(1)\right]$

if and only if $\ln(a^3+1) = \ln\left[(8a^3+1)^{\frac{1}{2}}\right]$ iff $(a^3+1)^2 = 8a^3+1$

iff $a^6 - 6a^3 = 0$ iff $a^3(a^3-6) = 0$ iff $a = 0$ or $a = \sqrt[3]{6}$.

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