Mathematics 170
Tutorial Worksheet #4
Solutions

1. (a) $\frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{3}{2}x^{\frac{1}{2}}$ so $\frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}}\right) = \frac{2}{3}\cdot\frac{3}{2}x^{\frac{1}{2}} = \sqrt{x}$.

$\therefore \int_{25}^{36}\sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_{25}^{36} = \frac{2}{3}\left(36^{\frac{3}{2}} - 25^{\frac{3}{2}}\right) = \frac{2}{3}(6^3-5^3) = \frac{2}{3}(216-125) = \frac{182}{3}$

(b) $\int_2^8 \sqrt{2x}\,dx = \int_2^8 \sqrt{2}\sqrt{x}\,dx = \sqrt{2}\int_2^8\sqrt{x}\,dx = \sqrt{2}\left[\frac{2}{3}x^{\frac{3}{2}}\Big|_2^8\right]$

$= \frac{2\sqrt{2}}{3}\left(2^{\frac{9}{2}} - 2^{\frac{3}{2}}\right) = \frac{2\sqrt{2}}{3}\cdot\sqrt{2}\left(2^4 - 2^1\right) = \frac{4}{3}(16-2) = \frac{56}{3}$

(c) $\frac{d}{dx}(3x+4)^{\frac{3}{2}} = \frac{3}{2}(3x+4)^{\frac{1}{2}}\cdot 3 = \frac{9}{2}\sqrt{3x+4}$, so $\frac{d}{dx}\left[\frac{2}{9}(3x+4)^{\frac{3}{2}}\right]$

$= \sqrt{3x+4}$

$\therefore \int_0^{20}\sqrt{3x+4}\,dx = \frac{2}{9}(3x+4)^{\frac{3}{2}}\Big|_0^{20} = \frac{2}{9}\left[64^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$

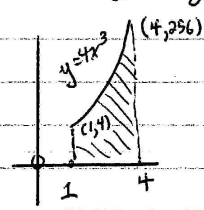
$= \frac{2}{9}(512-8) = \frac{2}{9}(504) = 2\times 56 = 112$

(d) $\int_0^{\frac{\pi}{4}}\left(3\cos(t) - 2\sec^2(t)\right)dt = 3\int_0^{\frac{\pi}{4}}\cos(t)\,dt - 2\int_0^{\frac{\pi}{4}}\sec^2(t)\,dt$

$= 3\left(\sin t\Big|_0^{\frac{\pi}{4}}\right) - 2\left(\tan(t)\Big|_0^{\frac{\pi}{4}}\right) = 3\left(\frac{1}{\sqrt{2}} - 0\right) - 2(1-0)$

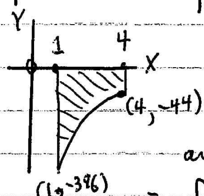
$= \frac{3\sqrt{2}}{2} - \frac{4}{2} = \frac{3\sqrt{2}-4}{2}$.

2. (a)



area $= \int_1^4 4x^3\,dx = x^4\Big|_1^4 = 256-1 = 255$ square units.

(b)

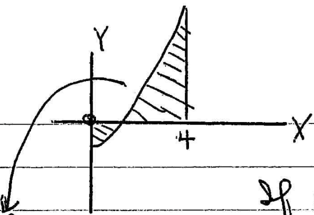


As the region lies below the X-axis, the area of the region is the negative of the integral

area $= -\left[\int_1^4(4x^3-400)\,dx\right] = -\left[x^4 - 400x\Big|_1^4\right]$

$= -\left[(256-1600) - (1-400)\right]$

$= -[-1344+399] = 945$ sq. units

2. (c) 

$y=4x^3-4$

We see that $4x^3-4=0$ when $x=1$, so the curve crosses the X-axis at $(1,0)$.

If $0 \le x \le 1$ the region is below the X-axis, and its area is $-\int_0^1 (4x^3-4)\,dx$

If $1 \le x \le 4$ the region is above the X-axis, and the area of that portion of the region is $\int_1^4 (4x^3-4)\,dx$.

$\therefore$ the total area of the shaded region is $-\int_0^1 (4x^3-4)\,dx + \int_1^4 (4x^3-4)\,dx$

$= -\left[(x^4-4x)\Big|_0^1\right] + \left[(x^4-4x)\Big|_1^4\right] = -\left[-3\right] + \left[(256-16)-(1-4)\right]$

$= 3 + 240 + 3 = 246$ sq. units

3. (a) Let $F(x) = \int_0^x \sqrt{1+y^3}\,dy$. $\therefore F'(x) = \sqrt{1+x^3}$

$\therefore \dfrac{d}{dx}\left(\int_0^{x^2} \sqrt{1+y^3}\,dy\right) = \dfrac{d}{dx}\left(F(x^2)\right) = F'(x^2)\cdot\dfrac{d}{dx}(x^2)$ by the chain rule

$= \sqrt{1+(x^2)^3}\cdot 2x = \left(\sqrt{1+x^6}\right)(2x)$. When $x=\sqrt{2}$ this becomes

$\sqrt{1+(2^{\frac{1}{2}})^6}\cdot 2\sqrt{2} = \sqrt{1+2^3}\cdot 2\sqrt{2} = 3 \times 2\sqrt{2} = 6\sqrt{2}$.

(b) $\dfrac{d}{dx}\left(\int_{3x}^{1} \sin\left(\dfrac{\pi t^2}{2}\right)dt\right) = -\dfrac{d}{dx}\int_1^{3x} \sin\left(\dfrac{\pi t^2}{2}\right)dt$

$= -\left[\left(\sin\left(\dfrac{\pi(3x)^2}{2}\right)\right)\cdot(3)\right] = -3\sin\left(\dfrac{9\pi x^2}{2}\right)$

(c) $\dfrac{d}{dx}\left[\int_{2x+1}^{3x+2} e^{-t^2}\,dt\right] = \dfrac{d}{dx}\left[\int_{2x+1}^{0} e^{-t^2}\,dt + \int_0^{3x+2} e^{-t^2}\,dt\right]$

$= \dfrac{d}{dx}\left[\int_0^{3x+2} e^{-t^2}\,dt\right] - \dfrac{d}{dx}\left[\int_0^{2x+1} e^{-t^2}\,dt\right] = e^{-(3x+2)^2}\cdot 3 - e^{-(2x+1)^2}\cdot 2$

4. $\lim_{x \to 0} \dfrac{\int_0^x \ln(1+t)\,dt}{x^2}$ is a $0/0$ indeterminate form, with both numerator and denominator being continuous. So by L'Hopital's Rule, we have

$$\lim_{x \to 0} \frac{\int_0^x \ln(1+t)\,dt}{x^2} = \lim_{x \to 0} \left[\frac{\frac{d}{dx}\int_0^x \ln(1+t)\,dt}{\frac{d}{dx}(x^2)} \right]$$

$$= \lim_{x \to 0} \frac{\ln(1+x)}{2x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{2} = \frac{1}{1+0} \Big/ 2$$

another $0/0$ indeterminate form, so use L'Hopital's rule again

$$= \frac{1}{2}$$

5. (a) $\int \left(3x^5 - 2\sin x + \frac{1}{x}\right)dx = 3\int x^5\,dx + 2\int(-\sin x)\,dx + \int \frac{1}{x}\,dx$

$$= \frac{3x^6}{6} + 2\cos x + \ln|x| + C$$

(b) $\int \dfrac{(x^2+1)^3}{x}\,dx = \int \dfrac{(x^2)^3 + 3(x^2)^2(1) + 3(x^2)1^2 + 1}{x}\,dx$

$$= \int (x^5 + 3x^3 + 3x + 1)\,dx = \frac{x^6}{6} + \frac{3x^4}{4} + \frac{3x^2}{2} + x + C$$

6. $F'(x) = \dfrac{3x^2+1}{\sqrt{x}} = 3x^{\frac{3}{2}} + x^{-\frac{1}{2}}$, so $F(x) = \int \left(3x^{\frac{3}{2}} + x^{-\frac{1}{2}}\right)dx$

$$= \frac{3x^{\frac{5}{2}}}{5/2} + 2x^{\frac{1}{2}} + C$$ where C is a constant. Set $x=4$ and get

$$5 = F(4) = \frac{6}{5}\left(4^{\frac{5}{2}}\right) + 2\left(4^{\frac{1}{2}}\right) + C = \frac{192}{5} + 4 + C$$

so $C = \dfrac{25 - 192 - 20}{5} = \dfrac{-187}{5}$ $\therefore$ $F(1) = \dfrac{6(1^{\frac{5}{2}})}{5} + 2(1^{\frac{1}{2}}) - \dfrac{187}{5}$

$$= \frac{6 + 10 - 187}{5} = \frac{-171}{5} \quad \#$$