136.170 Jan - April 2006

Solutions to Tutorial 2 worksheet

1. Find $\frac{dy}{dx}$ for the following curves.

(a) $x = t^3 + 3t^2 - 9t + 1$, $y = t^2$ $(-\infty < t < \infty)$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} \quad \text{if } \frac{dx}{dt} = 0.$$

$$\frac{dy}{dt} = 2t \qquad \frac{dx}{dt} = 3t^2 + 6t - 9$$
$$= 3(t^2 + 2t - 3)$$
$$= 3(t+3)(t-1)$$

$\therefore \frac{dy}{dx} = \frac{2t}{3(t+3)(t-1)}$ unless $t = -3$ or $t = 1$ (at which places the denominator is zero and the curve is momentarily vertical. If $t = -3$, $x = -27 - 27 + 27 + 1 = -26$ and $y = 9$.

If $t = 1$, $x = 1 + 3 - 9 + 1 = -4$ and $y = 1$

$\therefore$ the curve is momentarily vertical at $(-26, 9)$ and at $(-4, 1)$.

(b) $x = \sqrt{t}$ $y = t - 1$, $0 \le t \le 4$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} \quad \frac{dy}{dt} = 1 \quad \therefore \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{1}{1/2\sqrt{t}} = 2\sqrt{t}$$

if $0 < t \le 4$ (Note $\frac{dx}{dt}$ doesn't exist at $t = 0$).

$\left(\frac{dy}{dx}\Big|_{t=0} = 0 \right.$ as $y = x^2 - 1$ if $0 \le t \le 4$).

2. $x = \sin(2t)$ $\quad y = \sin(t)$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{\cos(t)}{2\cos(2t)}$$ Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$, so

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{[2\cos(2t)][-\sin(t)] - [\cos t][-4\sin(2t)]}{4\cos^2(2t)}}{2\cos(2t)}$$

$$= \frac{-2\sin(t)\cos(2t) + 4\cos(t)\sin(2t)}{8\cos^3(2t)}$$ (except for those t for which $\cos(2t)=0$,

i.e. $2t = (2n+1)\frac{\pi}{2}$ where n is an integer,

i.e. $t = \frac{(2n+1)\pi}{4}$ " " " " (as these values of t

make $\cos(2t) = 0$).

3. $x = t^2 - t$ $\quad$ What is the equation of the
$y = t^2 + t$ $\quad$ tangent line at the point where $t = 2$?

If $t = 2$, $x = 2^2 - 2 = 2$ and $y = 2^2 + 2 = 6$
so $(2,6)$ is the point of tangency.

$$\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{dy}{dt} \Big/ \frac{dx}{dt}\right|_{t=2} = \left.\frac{2t-1}{2t+1}\right|_{t=2} = \frac{3}{5}$$ (slope of tangent line)

$\therefore$ the tangent line has equation $\frac{y-6}{x-3} = \frac{3}{5}$, or $y = \frac{3x}{5} + \frac{12}{5}$.

4. (a) $x = t^3 + 3t^2 - 9t + 1$ $(-\infty < t < \infty)$
$y = t^2$.

Where are the tangent lines horizontal? Vertical?

$$\frac{dx}{dt} = 3t^2 + 6t - 9 \qquad \frac{dy}{dt} = 2t$$

If $t = 0$, $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = -9$ so $\frac{dy}{dx} = \frac{0}{-9} = 0$ and we have a horizontal tangent line at the point $(1, 0)$ as $x(0) = 1$ and $y(0) = 0$.

Candidates for ~~vertical~~ vertical tangent lines occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ (as $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$) Now $\frac{dx}{dt} = 3t^2 + 6t - 9$
$= 3(t^2 + 2t - 3)$
$= 3(t+3)(t-1)$, so $\frac{dx}{dt} = 0$ when $t = 1$ and $t = -3$ (and $\frac{dy}{dt} \neq 0$ when $t = 1$ or -3). $\therefore$ The curve is vertical when $t = 1$, i.e. at $(-5, 1)$ and when $t = -3$, i.e. at $(28, 9)$.

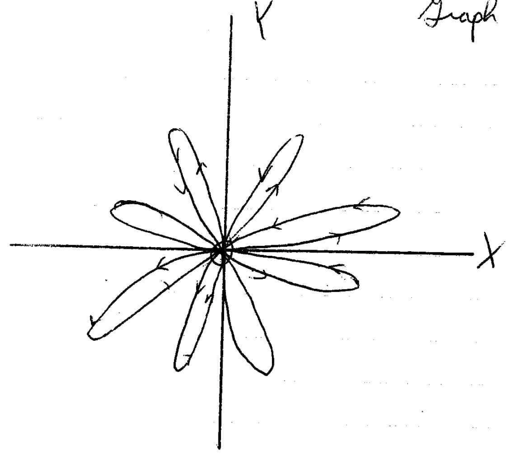
(b) $x = \sqrt{t}$ $y = t - 1$ $(0 \leq t \leq 4)$

From 1(b), $\frac{dy}{dx} = 2\sqrt{t}$ if $0 < t \leq 4$
and $\lim_{t \to 0^+} \frac{dy}{dx} = 0$. $\therefore$ There is a horizontal tangent line at $(0, -1)$ and no vertical tangent lines.

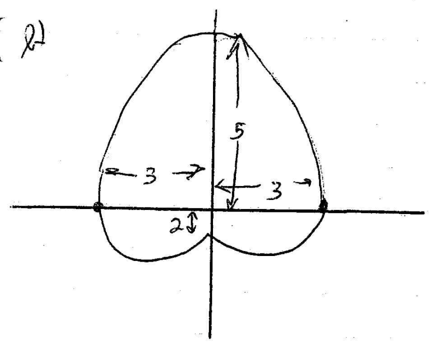
5. (a)

Graph of $r = \sin(4\theta)$

(very rough)



(b)



$r = 3 + 2\sin\theta$

θ	r
0	3
$\frac{\pi}{4}$	$3 + \frac{2}{\sqrt{2}}$
$\frac{\pi}{2}$	5 (maximum r)
$\frac{3\pi}{4}$	$3 + \sqrt{2}$
π	3
$\frac{5\pi}{4}$	$3 - \frac{2}{\sqrt{2}}$
$\frac{3\pi}{2}$	2 (minimum r)
$\frac{7\pi}{4}$	$3 - \frac{2}{\sqrt{2}}$
2π	3

6 (a) $r = \sin(4\theta)$

$x = r\cos(\theta) = \sin(4\theta)\cos\theta$

$y = r\sin(\theta) = \sin(4\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin(4\theta)\cos\theta + 4\cos(4\theta)\sin\theta}{-\sin(4\theta)\sin\theta + 4\cos(4\theta)\cos\theta}$$

When $\theta = \frac{\pi}{6}$, this is $\dfrac{\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) + 4\cos\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)}{-\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) + 4\cos\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{6}\right)}$

$$= \frac{\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + 4\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}{-\frac{\sqrt{3}}{2} \cdot \frac{1}{2} + 4\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} = \frac{\frac{3}{2} - 1}{-\frac{\sqrt{3}}{4} - \sqrt{3}} = \frac{\frac{1}{2}}{-\frac{5\sqrt{3}}{4}}$$

$$= -\frac{2\sqrt{3}}{15}$$

(b) $r = 3 + 2\sin(\theta)$

$x = r\cos\theta = 3\cos\theta + 2\sin\theta\cos\theta = 3\cos\theta + \sin(2\theta)$

$y = (3 + 2\sin\theta)\sin\theta = 3\sin\theta + 2\sin^2\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta - 4\sin\theta\cos\theta}{-3\sin\theta + 2\cos(2\theta)}$$

$$= \frac{3\cos\theta - 2\sin(2\theta)}{-3\sin\theta + 2\cos(2\theta)}$$

When $\theta = \frac{\pi}{6}$ we get

$$\frac{dy}{dx} = \frac{3\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)}{-3\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

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