Math 170

Problem Worksheet 11 Solutions

1. Do these converge? If so, to what?

(a) $\int_0^2 \frac{1}{x^{\frac{1}{3}}}\,dx$. This is improper because $\frac{1}{x^{\frac{1}{3}}}$ is not defined at 0. We compute:

$$\lim_{r\to 0^+} \int_r^2 \frac{1}{x^{\frac{1}{3}}}\,dx = \lim_{r\to 0^+} \int_r^2 x^{-\frac{1}{3}}\,dx = \lim_{r\to 0^+} \left[\frac{3x^{\frac{2}{3}}}{2} \Big|_r^2 \right]$$

$$= \lim_{r\to 0^+} \left[\frac{3(2^{2/3})}{2} - \frac{3(r^{\frac{2}{3}})}{2} \right] = \frac{3(2^{\frac{2}{3}})}{2} - 0 = \frac{3(2^{\frac{2}{3}})}{2}$$

$= \frac{3}{\sqrt[3]{2}}$, so this integral converges to $\frac{3}{\sqrt[3]{2}}$.

(b) $\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx$. $4-x^2>0$ if $1\le x<2$ but $4-x^2=0$ if $x=2$, so the integrand is undefined at $x=2$.

So we compute: $\lim_{r\to 0^+} \int_1^{2-r} \frac{1}{\sqrt{4-x^2}}\,dx$

Now if $x=2u$, then $dx=2du$ so $\int \frac{1}{\sqrt{4-x^2}}\,dx$

$$= \int \frac{2\,du}{\sqrt{4-4u^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) = \sin^{-1}\left(\frac{x}{2}\right)$$

So $\displaystyle\int_1^{2-r} \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right)\Big|_1^{2-r} = \sin^{-1}\left(1-\frac{r}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$

$$\therefore \lim_{r\to0^+} \int_1^{2-r} \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{r\to0^+}\left[\sin^{-1}\left(1-\frac{r}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

so this improper integral converges to $\frac{\pi}{3}$.

(c) $\displaystyle\int_{-1}^{0} \frac{1}{x(x+1)}\,dx$. The denominator is 0 both at $x=0$ and at $x=-1$.

$\therefore$ we examine $\displaystyle\int_{-1}^{-\frac{1}{2}} \frac{1}{x(x+1)}\,dx + \int_{-\frac{1}{2}}^{0} \frac{1}{x(x+1)}\,dx$.

Our given integral converges if and only if <u>both</u> of the above integrals converge. Now

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad \text{so } 1 = A(x+1) + Bx.$$

Set $x=0$ and get $A=1$
Set $x=-1$ and get $B=-1$

So $\displaystyle\int \frac{1}{x(x+1)}\,dx = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \ln|x| - \ln|x+1|$

$$\therefore \lim_{r\to0^-} \int_{-\frac{1}{2}}^{r} \frac{1}{x(x+1)}\,dx = \lim_{r\to0^-}\left[\ln|r| - \ln|r+1|\right]$$

But $\lim_{r\to 0^-} \ln|r| = -\infty$ so this integral diverges.

Thus the given integral diverges. [In fact $\lim_{r\to 0^+} \int_{-1+r}^{-\frac{1}{2}} \frac{1}{x(x+1)}\,dx$ also diverges. But showing that at least one of our two integrals diverges suffices.].

(d) $\int_0^3 \frac{e^x}{e^x-1}\,dx$. Now when $x=0$, $e^x-1 = e^0-1 = 1-1=0$ so the integrand $\to +\infty$ as $x\to 0^+$.

So we compute: let $u=e^x-1$. Then $du = e^x\,dx$

When $x=0$, $u=e^0-1=0$. When $x=3$, $u=e^3-1$

$\therefore$ our integral is $\int_0^{e^3-1} \frac{du}{u}$, and $\lim_{u\to 0^+} \frac{1}{u} = +\infty$ (just as $\lim_{x\to 0^+} \frac{e^x}{e^x-1} = +\infty$),

so we compute: $\lim_{\delta\to 0^+} \int_\delta^{e^3-1} \frac{du}{u} = \lim_{\delta\to 0^+}\left(\ln|u|\Big|_\delta^{e^3-1}\right)$

$$= \lim_{\delta\to 0^+}\big(\ln(e^3-1) - \underbrace{\ln \delta}_{\text{This} \to -\infty \text{ as } \delta\to 0^+}\big) = -\infty,$$ so our integral diverges.

2. (a) $\frac{x^2}{x^5+1}$ is a well-defined continuous function on $[0,1]$ so $\int_0^1 \frac{x^2}{x^5+1}\,dx$ exists. Hence $\int_0^\infty \frac{x^2}{x^5+1}\,dx$ will converge if $\int_1^\infty \frac{x^2}{x^5+1}\,dx$ converges.

But $\frac{x^2}{x^5+1} \le \frac{x^2}{x^5} = \frac{1}{x^3}$ if $x \ge 1$ and $\frac{1}{x^3}$ is continuous on $[1, \infty)$ Furthermore, $\int_1^\infty \frac{1}{x^3}\,dx = \lim_{R\to\infty}\left[-\frac{1}{2x^2}\Big|_1^R\right]$

$= \lim_{R\to\infty}\left(-\frac{1}{2R^2} + \frac{1}{2}\right) = \frac{1}{2}$, so $\int_1^\infty \frac{1}{x^3}\,dx$ converges. $\therefore$ by the Comparison Test, $\int_1^\infty \frac{x^2}{x^5+1}\,dx$ converges, and so $\int_0^\infty \frac{x^2}{x^5+1}\,dx$ converges. [Note that "$\frac{x^2}{x^5+1} \le \frac{1}{x^3}$" is meaningless at $x=0$ as $\frac{1}{x^3}$ is not defined at $x=0$, so we cannot use the comparison test immediately on the interval $[0, +\infty)$].

(f) $\int_2^\infty \frac{x\sqrt{x}}{x^2-1}\,dx$. Now $0 < x^2-1 < x^2$ if $x \ge 2$ so

$$\frac{x\sqrt{x}}{x^2-1} \ge \frac{x\sqrt{x}}{x^2} = \frac{1}{\sqrt{x}} \text{ if } x \ge 2.$$ But

$$\int_2^\infty \frac{dx}{\sqrt{x}} = \lim_{R\to\infty}\int_2^R \frac{dx}{\sqrt{x}} = \lim_{R\to\infty}\left[2\sqrt{x}\Big|_2^R\right] = \lim_{R\to\infty}\left[2\sqrt{R} - 2\sqrt{2}\right] = +\infty$$

so this integral diverges. But $\frac{x\sqrt{x}}{x^2-1} \ge \frac{1}{\sqrt{x}}$ if $x \ge 2$, so by the comparison test, $\int_2^\infty \frac{x\sqrt{x}}{x^2-1}\,dx$ also diverges.

(g) $\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$ Both $\int_0^1 \frac{dx}{\sqrt{x}+x^2}$ and $\int_1^\infty \frac{dx}{\sqrt{x}+x^2}$ are improper integrals (note that $\frac{1}{\sqrt{x}+x^2}$ is undefined at 0), so each of these must converge if the given function is to converge.

Also observe that $\sqrt{x}+x^2 = \sqrt{x}(1+x^{\frac{3}{2}})$.

Now $1+x^{\frac{3}{2}} \geq 1$ if $0 \leq x \leq 1$ so $\frac{1}{\sqrt{x}} \geq \frac{1}{\sqrt{x}(1+x^{\frac{3}{2}})}$ if $0 \leq x \leq 1$.

But $\lim_{c \to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c \to 0^+} \left[2\sqrt{x} \Big|_c^1 \right] = \lim_{c \to 0^+} (2 - 2\sqrt{c}) = 2$

$\therefore$ by the Comparison Test, $\int_0^1 \frac{1}{\sqrt{x}(1+x^{\frac{3}{2}})}\,dx = \int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx$ converges. Finally, observe that if $1 \leq x$ then

$\frac{1}{\sqrt{x}+x^2} \leq \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so by the Comparison Test, $\int_1^\infty \frac{dx}{\sqrt{x}+x^2}$ converges. Putting our results together, we see that $\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$ converges.

(k) $\int_2^\infty \frac{dx}{\sqrt{x}\ln(x)}$ If $x \geq 2$ then $\sqrt{x} \leq x$ so

$\frac{1}{\sqrt{x}} \geq \frac{1}{x}$ so $\frac{1}{\sqrt{x}\ln(x)} \geq \frac{1}{x\ln(x)}$.

Now $\int \frac{dx}{x\ln(x)} = \ln(|\ln(x)|)$ (Let $u = \ln(x)$, so $du = \frac{1}{x}dx$, so $\int \frac{dx}{x\ln(x)} = \int \frac{1}{u}du = \ln|u| = \ln(|\ln(x)|)$)

$$\therefore \int_2^R \frac{dx}{x\ln(x)} = \ln(|\ln(x)|)\Big|_2^R$$

$$= \ln(\ln R) - \ln(\ln 2)$$

As $R \to +\infty$, $\ln R \to +\infty$ so $\ln(\ln R) \to +\infty$ so $\lim_{R \to \infty} \int_2^R \frac{dx}{x\ln(x)} = +\infty$

and $\int_2^\infty \frac{dx}{x\ln(x)}$ diverges. $\therefore$ by the Comparison Test, $\int_2^\infty \frac{dx}{\sqrt{x}\ln(x)}$ diverges.

3. (a) By the distance formula, the distance along $y = ax + b$ from $(A, aA + b)$ to $(B, aB + b)$ is

$$\sqrt{(A-B)^2 + [(aA+b) - (aB+b)]^2} = \sqrt{(A-B)^2 + a^2(A-B)^2}$$

$$= |A-B|\sqrt{1+a^2} = (B-A)\sqrt{1+a^2}$$ if $B > A$

Using the arc length formula, $y = ax + b$

so $\frac{dy}{dx} = a$ so Length $= \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_A^B \sqrt{1+a^2}\, dx = \sqrt{1+a^2}\, x \Big|_A^B = (B-A)\sqrt{1+a^2}.$$

(b) $y^2 = (x-1)^3$ from $(1,0)$ to $(2,1)$

$y = (x-1)^{\frac{3}{2}}$ so $\frac{dy}{dx} = \frac{3}{2}(x-1)^{\frac{1}{2}}$ so

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{9}{4}(x-1)} = \sqrt{\frac{4 + 9x - 9}{4}} = \frac{\sqrt{9x-5}}{2}$$

for $1 \le x \le 2$ $\therefore$ length $= \int_1^2 \frac{1}{2}(9x-5)^{\frac{1}{2}}\, dx = \frac{1}{18} \frac{(9x-5)^{\frac{3}{2}}}{3/2} \Big|_1^2$

$$= \frac{1}{27}\left[(18-5)^{\frac{3}{2}} - (9-5)^{\frac{3}{2}}\right] = \frac{1}{27}\left[13^{\frac{3}{2}} - 8\right]$$