Mathematics 170

Problem Worksheet No 10

Wed. March 23 – Tues March 29

SOLUTIONS

1. (a) $$\frac{x}{x^3+1} = \frac{x}{(x+1)(x^2-x+1)}$$ $x^2-x+1 = \left(x-\frac{1}{2}\right)^2+\frac{3}{4}$, so x^2-x+1 is an irreducible quadratic form. Hence

$$\frac{x}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \quad \cdots (*)$$

, where A, B, and C are constants.

Multiply $(*)$ through by x^3+1 and get.

$$x = A(x^2-x+1)+(Bx+C)(x+1) \quad \cdots\cdots (**)$$

Put $x=-1$ in $(**)$ and get:

$$-1 = 3A \quad \text{or} \quad A = -\frac{1}{3}$$

Put $x=0$ in $(**)$ and get:

$$0 = -\frac{1}{3} + C \quad \text{or} \quad C = \frac{1}{3}$$

Put $x=1$ in $(**)$ and get $1 = -\frac{1}{3} + \left(B+\frac{1}{3}\right)(1+1)$

$\frac{4}{3} = 2\left(B+\frac{1}{3}\right)$, so $B = \frac{1}{3}$. Thus

$$\frac{x}{x^3+1} = -\frac{1}{3(x+1)} + \frac{\frac{1}{3}x+\frac{1}{3}}{x^2-x+1}$$

$$\therefore \int_0^1 \frac{x}{x^3+1}\,dx = \frac{1}{3}\left[-\int_0^1 \frac{dx}{x+1} + \int_0^1 \frac{x+1}{x^2-x+1}\,dx\right] \quad \cdots (***)$$

Now $\frac{d}{dx}(x^2-x+1) = 2x-1$ so we write

$$\frac{x+1}{x^2-x+1} = \frac{1}{2}\,\frac{(2x-1)+3}{x^2-x+1} = \frac{1}{2}\left[\frac{2x-1}{x^2-x+1}\right] + \frac{3}{2}\left(\frac{1}{x^2-x+1}\right)$$

So $(***)$ becomes

$$\int_0^1 \frac{x}{x^3+1}\,dx = \frac{1}{3}\left[-\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{2x-1}{x^2-x+1}\,dx + \frac{3}{2}\int_0^1 \frac{dx}{x^2-x+1}\right] \quad \cdots (+)$$

Let $u = x^2-x+1$
$\therefore du = (2x-1)\,dx$

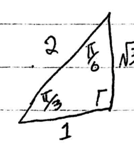
When $x=0$, $u=1$
When $x=1$, $u=1$

$$\therefore \int_0^1 \frac{2x-1}{x^2-x+1}\,dx = \int_1^1 \frac{du}{u} = \ln|u|\Big|_1^1 = 0-0 = 0.$$

Also, $\int_0^1 \frac{dx}{x+1} = \ln|x+1|\Big|_0^1 = \ln 2 - \ln 1 = \ln 2$. Substituting into $(+)$, we see that

$$\int_0^1 \frac{x}{x^3+1}\,dx = -\frac{1}{3}\ln 2 + \frac{1}{2}\int_0^1 \frac{dx}{x^2-x+1} \quad \cdots (++)$$

To evaluate $\int_0^1 \frac{dx}{x^2-x+1}$ note that $x^2-x+1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}$



So, let $x-\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$. $\therefore dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$ and

$$\left(x-\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\tan^2\theta + \frac{3}{4} = \frac{3}{4}\sec^2\theta.$$

When $x=0$, $\tan\theta = -\frac{1}{\sqrt{3}}$
so $\theta = -\frac{\pi}{6}$

and when $x = 1$, $\tan\theta = \frac{1}{\sqrt{3}}$ so $\theta = \frac{\pi}{6}$

Thus $\displaystyle\int_0^1 \frac{dx}{x^2 - x + 1} = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta}{\frac{3}{4}\sec^2\theta} = \frac{2\sqrt{3}}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\theta$

$= \frac{2\sqrt{3}}{3} \cdot \frac{\pi}{3} = \frac{2\sqrt{3}\,\pi}{9}$ Thus by (++),

$$\int_0^1 \frac{x}{x^3+1}\,dx = -\frac{1}{3}\ln 2 + \frac{1}{2}\left(\frac{2\sqrt{3}\,\pi}{9}\right) = \frac{-3\ln 2 + \sqrt{3}\,\pi}{9}$$

1. (b) As x^3+1 and x^3+x both have degree 3, divide x^3+x into x^3+1. We do this indirectly:

$$\frac{x^3+1}{x^3+x} = \frac{x^3+x}{x^3+x} - \frac{x-1}{x^3+x} = 1 - \frac{x-1}{x(x^2+1)} \quad \ldots (*)$$

Now $\displaystyle\frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ so

$x - 1 = Ax^2 + A + Bx^2 + Cx$. Set $x = 0$ and get

$A = -1$ Set $x=1$ and get $0 = 2A + B + C$

$0 = -2 + B + C$

Set $x = -1$ and get $-2 = -2 + B - C$, or $B = C$

$\therefore 2B = 2$ so $B = 1$ and $C = 1$ Thus

$$\frac{x-1}{x(x^2+1)} = \frac{-1}{x} + \frac{x+1}{x^2+1}$$ Hence by (*) above we get:

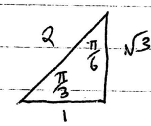
$$= -\frac{1}{x} + \frac{1}{2}\left(\frac{2x}{x^2+1}\right) + \frac{1}{x^2+1}$$

$$\int_1^{\sqrt{3}} \frac{x^3+1}{x^3+x}\,dx = \int_1^{\sqrt{3}} 1\,dx + \int_1^{\sqrt{3}} \frac{1}{x}\,dx - \frac{1}{2}\int_1^{\sqrt{3}} \frac{2x}{x^2+1}\,dx - \int_1^{\sqrt{3}} \frac{dx}{x^2+1}$$

$$= x\Big|_1^{\sqrt{3}} + \ln|x|\Big|_1^{\sqrt{3}} - \frac{1}{2}\ln(x^2+1)\Big|_1^{\sqrt{3}} - \tan^{-1}(x)\Big|_1^{\sqrt{3}}$$

$$= \sqrt{3}-1 + \ln(\sqrt{3}) - \ln 1 - \frac{1}{2}\ln 10 + \frac{1}{2}\ln 2 - \tan^{-1}(\sqrt{3}) + \tan^{-1} 1$$

$$= \sqrt{3}-1 + \ln\left(\sqrt{\frac{3}{5}}\right) - \frac{\pi}{3} + \frac{\pi}{4}$$



$$= \sqrt{3}-1 + \ln\left(\sqrt{\frac{3}{5}}\right) - \frac{\pi}{12}$$

1 (c) $\displaystyle\int_3^8 \frac{x^2}{1+\sqrt{x+1}}\,dx.$ Let $u = 1+\sqrt{x+1}$

$\therefore (u-1)^2 = x+1$

$x = u^2 - 2u \qquad \frac{dx}{du} = 2u-2 \qquad dx = (2u-2)\,du$

When $x=3$, $u = 1+\sqrt{4} = 3$. When $x=8$, $u = 1+\sqrt{9} = 4$

So our integral becomes $\displaystyle\int_3^4 \frac{(u^2-2u)^2\,(2u-2)\,du}{u}$

$$= \int_3^4 \frac{(u^4 - 4u^3 + 4u^2)(2)(u-1)\,du}{u} = 2\int_3^4 (u^4 - 5u^3 + 8u^2 - 4u)\,du$$

$$= 2\left(\frac{u^5}{5} - \frac{5u^4}{4} + \frac{8u^3}{3} - 2u^2 \Big|_3^4\right) = 2u^2\left[\frac{u^3}{5} - \frac{5u^2}{4} + \frac{8u}{3} - 2\Big|_3^4\right] = \frac{1327}{30}$$

2. (a) $\int x e^{-2x}\,dx$: Let $u = x$ $dv = e^{-2x}\,dx$

$\therefore du = dx$ $v = \int e^{-2x}\,dx = -\frac{e^{-2x}}{2}$

$$= uv - \int v\,du = -\frac{xe^{-2x}}{2} - \int\left(-\frac{e^{-2x}}{2}\right)dx$$

$$= -\frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx$$

$$= -\frac{xe^{-2x}}{2} - \frac{1}{4}e^{-2x} = -\left[\frac{e^{-2x}(2x+1)}{4}\right]$$

$$\therefore \int_0^R x e^{-2x}\,dx = -\frac{1}{4}\left(e^{-2x}\right)(2x+1)\Big|_0^R = -\frac{1}{4}\left(e^{-2R}\right)(2R+1) + \frac{1}{4}(1)(+1)$$

$$= \frac{1}{4} - \frac{2R+1}{4e^R}$$

So $\int_0^\infty x e^{-2x}\,dx = \lim_{R\to+\infty} \frac{1}{4} - \frac{2R+1}{4e^R} = \frac{1}{4} - \lim_{R\to+\infty}\frac{2R+1}{4e^R}$

$= \frac{1}{4} - \lim_{R\to\infty}\frac{2}{4e^R} = \frac{1}{4} - 0 = \frac{1}{4}$ so this integral converges to $1/4$.

by L'Hopital's rule

2. (b) $\frac{x}{x^2+4x+5} = \frac{x}{(x+2)^2+1}$. Let $x+2 = \tan\theta$

$\therefore dx = \sec^2\theta\,d\theta$

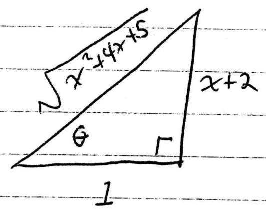
and $(x+2)^2+1 = \tan^2\theta+1$

$= \sec^2\theta$

Then $\displaystyle\int \frac{x}{x^2+4x+5}\,dx = \int \frac{(\tan\theta - 2)(\sec^2\theta\, d\theta)}{\sec^2\theta}$

$\displaystyle = \int \frac{\sin\theta}{\cos\theta}\,d\theta - \int 2\,d\theta = \ln|\cos\theta| - 2\theta$

$\displaystyle = \ln\left(\frac{1}{\sqrt{(x+1)^2+1}}\right) - 2\tan^{-1}(x+2)$



[We see $\cos\theta = \dfrac{1}{\sqrt{x^2+4x+5}} = \dfrac{1}{\sqrt{(x+1)^2+1}}$ by consulting the picture of an angle θ whose tangent is $x+2$]

$\displaystyle = \frac{1}{2}\ln\left(\frac{1}{x^2+4x+5}\right) - 2\tan^{-1}(x+2)$

$\displaystyle = -\frac{1}{2}\ln(x^2+4x+5) - 2\tan^{-1}(x+2)$

Thus $\displaystyle\int_R^1 \frac{x}{x^2+4x+5}\,dx = \left(-\frac{1}{2}\ln 10 - 2\tan^{-1}3\right) + \frac{1}{2}\ln((R+1)^2+1) + 2\tan^{-1}(R+2).$

Now $\displaystyle\lim_{R\to-\infty} \ln((R+1)^2+1) = +\infty$, so this integral diverges.

2.(c) To compute $\int f'(2x)\,dx$, let $u = 2x$ $\therefore$ $dx = \dfrac{du}{2}$ so

$\displaystyle\int f'(2x)\,dx = \frac{1}{2}\int f'(u)\,du = \frac{f(u)}{2} = \frac{1}{2}\,\frac{1}{(2x)^2+1} = \frac{1}{2(4x^2+1)}$

$\displaystyle\therefore \lim_{R\to\infty}\int_0^R f'(2x)\,dx = \lim_{R\to\infty} \left.\frac{1}{2(4x^2+1)}\right|_0^R = \lim_{R\to\infty} \frac{1}{2(4R^2+1)} - \frac{1}{2} = \frac{1}{2}$

and $\displaystyle\lim_{R\to-\infty}\int_R^0 f'(2x)\,dx = \lim_{R\to-\infty} \frac{1}{2} - \frac{1}{2(4R^2+1)} = \frac{1}{2}.$ $\therefore$ the integral converges to $-\frac{1}{2} + \frac{1}{2} = 0.$