DEPARTMENT & COURSE NO: 136.170 TIME: 1 hour

EXAMINATION: Calculus 2 EXAMINER: Various

NAME: (Print in ink) ______________________________

STUDENT NUMBER: ______________

SIGNATURE: (in ink) ______________________________

(I understand that cheating is a serious offense)

Please indicate your instructor and section by checking the appropriate box below:

☐	L03	R. G. Woods	M, W, F	9:30am - 10:20am
☐	L04	S. Kalajdzievski	M, W, F	11:30am - 12:20pm
☐	L05	C. Podder	T, Th	1:00pm - 2:15pm
☐	L92	Challenge for Credit		

INSTRUCTIONS TO STUDENTS:

This is a 1 hour exam. Please show your work **clearly**.

No texts, notes,calculators, cell phones or other aids are permitted.

This exam has a title page, 3 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but do not remove the staples from the exam paper.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 40 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but please indicate clearly your work is continued.

Question	Points	Score
1	8	
2	7	
3	7	
4	5	
5	5	
6	3	
7	5	
Total:	40	

Brief Solutions

1. Use l'Hôpital's rule to evaluate

[3] (a) $\lim_{x \to 0} \dfrac{sin^2(x)}{1 - cos(x)}$

Solution First notice that both the numerator and the denominator tend to 0 as $x \to 0$. So, we can apply l'Hôpital's rule.

$$\lim_{x \to 0} \frac{sin^2(x)}{1 - cos(x)} = \lim_{x \to 0} \frac{2sin(x)cos(x)}{sin(x)} = \lim_{x \to 0} 2cos(x) = 2$$

[5] (b) $\lim_{x \to 0^+} (1 - x)^{1/x}$

Solution Set $y = (1 - x)^{\frac{1}{x}}$. So $\ln y = \ln((1 - x)^{\frac{1}{x}}) = \frac{1}{x}\ln(1 - x)$. We now compute

$$lim_{x \to 0^+} \ln y = lim_{x \to 0^+} \frac{1}{x}\ln(1 - x) = lim_{x \to 0^+} \frac{\ln(1 - x)}{x} =$$

$$= lim_{x \to 0^+} \frac{\frac{-1}{1-x}}{1} = -1.$$

where in the second to the last step we have used l'Hôpital's rule.

Since $lim_{x \to 0^+} \ln y = -1$ and since $lim_{x \to 0^+} \ln y = \ln(lim_{x \to 0^+} y)$, it follows that $\ln(lim_{x \to 0^+} y) = -1$ and so $lim_{x \to 0^+} y = e^{-1}$.

[7] 2. Consider the curve C with parametric equations

$$x(t) = t^3 - 12t, \quad y(t) = t^2 + 1 \qquad (-4 \leq t \leq 4)$$

(a) Find the equation of the tangent line to the curve at the point on the curve corresponding to $t = 1$.

Solution

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2 - 12}.$$

At $t = 1$ we get $\frac{dy}{dx} = -\frac{2}{9}$. This the slope of the tangent line. Since that line passes through the point $(-11, 2)$ (which happens on the given curve when $t = 1$), it follows that the tangent line we want has an equation $\frac{y-2}{x+11} = -\frac{2}{9}$.

(b) Give the coordinates of the point(s) on C at which C has a vertical tangent line.

Solution Tangents line are vertical where $\frac{dx}{dt} = 0$ while at the same time $\frac{dy}{dt} \neq 0$. Since $\frac{dx}{dt} = 3t^2 - 12$, that happens when $t = 2$ and when $t = -2$. These two moments yield the points $(-16, 5)$ and $(16, 5)$ respectively.

[4] 3. (a) Sketch the curve $r = 1 + \cos\theta$ for $0 \leq \theta \leq 2\pi$.

Solution

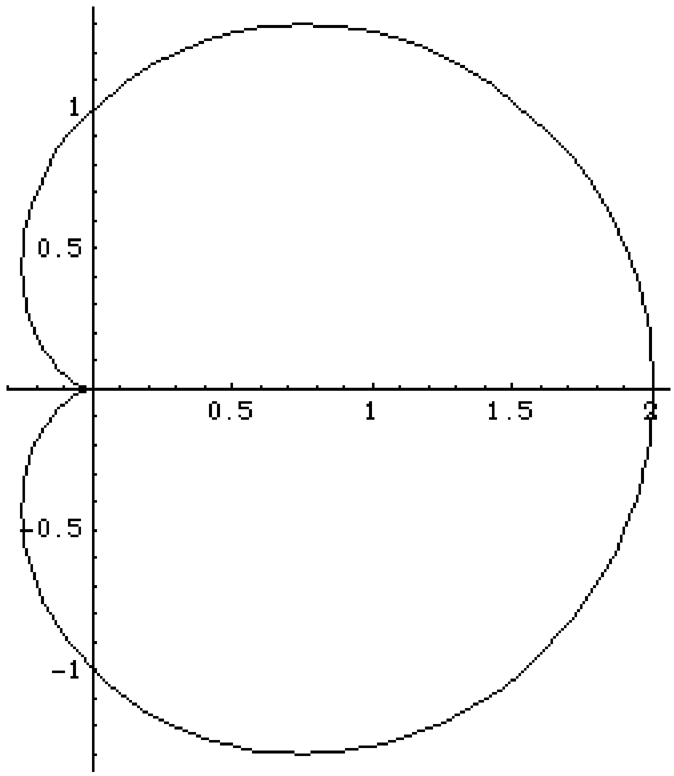

Figure 1: default

[3] (b) Find a polar equation for the curve whose equation in Cartesian coordinates is $x^2 - y^2 = 1$. Find the Cartesian coordinate(s) of the point(s) on this curve whose θ- coordinate is 0.

Solution Substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to get $r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1$. When $\theta = 0$ we have $r^2\cos^2(0) - r^2\sin^2(0) = 1$ which in turn gives $r = 1$ or $r = -1$. At $r = 1$ (and $\theta = 0$) we compute $x = 1$ and $y = 0$. At $r = -1$ (and $\theta = 0$) we compute $x = -1$ and $y = 0$. So we get the points $(1, 0)$ and $(-1, 0)$.

[5] 4. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int_{3x}^{5x} \frac{\mathrm{d}t}{15+t^4}\right)$ at $x=1$. Do not simplfy your answer.

Solution $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int_{3x}^{5x} \frac{\mathrm{d}t}{15+t^4}\right) = \dfrac{1}{15+(5x)^4}\cdot 5 - \dfrac{1}{15+(3x)^4}\cdot 3.$
When $x=1$ we get $\frac{1}{15+(5)^4}\cdot 5 - \frac{1}{15+(3)^4}\cdot 3.$

[5] 5. Evaluate the definite integral $\displaystyle\int_0^{\pi} \left(1+\sec^2\left(\frac{x}{4}\right)\right)\mathrm{d}x.$

Solution

$$\int_0^{\pi} \left(1+\sec^2\left(\frac{x}{4}\right)\right)\mathrm{d}x = \int_0^{\pi} 1\mathrm{d}x + \int_0^{\pi} \sec^2\left(\frac{x}{4}\right)\mathrm{d}x = x \,\|_0^{\pi} + 4\tan(\frac{x}{4})\,\|_0^{\pi} = \pi - 4$$

[3] 6. Evaluate the indefinite integral $\displaystyle\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 \mathrm{d}x.$

Solution

$$\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 \mathrm{d}x = \int \left(x-2+\frac{1}{x}\right)^2 \mathrm{d}x = \frac{x^2}{2} - 2x + \ln|x| + c$$

[5] 7. If $f(x)=x^3+1$, evaluate $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2}{n}\right) f\left(\frac{2i}{n}\right)$ by writing it as a definite integral with lower limit of integration 0, and then evaluating that integral. Briefly explain what you are doing.

Solution

If the upper limit of the integral is b then $\frac{2}{n} = \frac{b-0}{n}$ gives $b=2$. The left-hand point of the i-th subinterval is $\frac{2i}{n}$. With that we have that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2}{n}\right) f\left(\frac{2i}{n}\right) = \int_0^{\pi} f(x)\mathrm{d}x = \int_0^{\pi}(x^3+1)\mathrm{d}x = \frac{x^4}{4}+x \;\;|_0^2 = 6$$