[8] 1. (a) Find the points of the intersection of the curves $y=2x^2$ and $y=3-x^2$.

They intersect at points whose y-co-ordinates satisfy

$2x^2 = 3-x^2$

$3x^2 = 3$

$x^2 = 1$

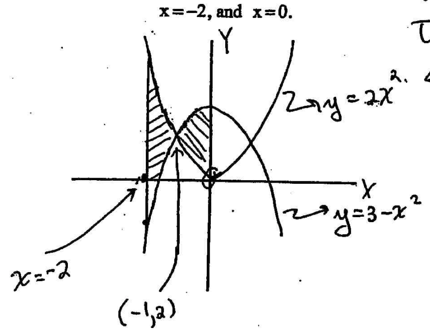
$x = \pm 1$

If $x=1$, $y=2(1)^2=2$

If $x=-1$, $y=2(-1)^2=2$

$\therefore$ the points of intersection are $(1,2)$ and $(-1,2)$

(b) Find the area of the region enclosed by the curves $y=2x^2$, $y=3-x^2$, $x=-2$, and $x=0$.



The region is the (shaded) area shown.

If $-2 \le x \le -1$ then $2x^2 \ge 3-x^2$.

If $-1 \le x \le 0$ then $3-x^2 \ge 2x^2$.

Thus the area A of the region is given by..

$$A = \int_{-2}^{-1} \left[2x^2-(3-x^2)\right] dx + \int_{-1}^{0} \left[(3-x^2)-(2x^2)\right] dx$$

$$= \int_{-2}^{-1} (3x^2-3)\, dx + \int_{-1}^{0} (3-3x^2)\, dx$$

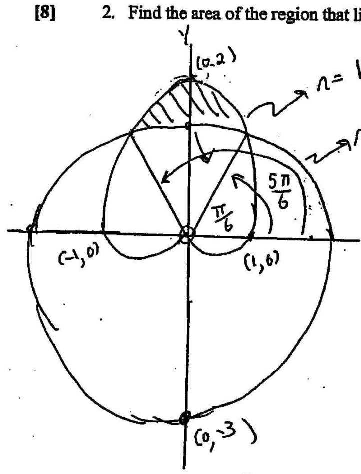
$$= (x^3-3x)\Big|_{-2}^{-1} + (3x-x^3)\Big|_{-1}^{0}$$

$$= \left[(-1+3)-(-8+6)\right] + \left[(0-0)-(-3-(-1))\right]$$

$$= (2+2) + (2) = 6 \text{ square units.}$$

continued...

[8] 2. Find the area of the region that lies inside $r=1+\sin\theta$ and outside $r=2-\sin\theta$.



[Co-ordinates plotted are Cartesian co-ordinates.]

The region under consideration has been shaded.

We see that

$1+\sin\theta = 2-\sin\theta$

if $2\sin\theta = 1$, i.e. $\sin\theta = \frac{1}{2}$,

i.e. $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

If A is the area of the shaded region, we see that

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2\,d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(2-\sin\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[\left(1+2\sin\theta+\sin^2\theta\right) - \left(4-4\sin\theta+\sin^2\theta\right)\right] d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (6\sin\theta - 3)\,d\theta$$

$$= \frac{1}{2}\left[-6\cos\theta - 3\theta \Big|_{\pi/6}^{5\pi/6}\right] = \frac{1}{2}\left[\left((-6)\left(-\frac{\sqrt{3}}{2}\right) - \frac{5\pi}{2}\right) - \left((-6)\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left(3\sqrt{3} + 3\sqrt{3} - 2\pi\right) = 3\sqrt{3} - \pi \text{ sq. units.}$$

continued...

March 11, 2005 5:30 pm - 6:30 pm

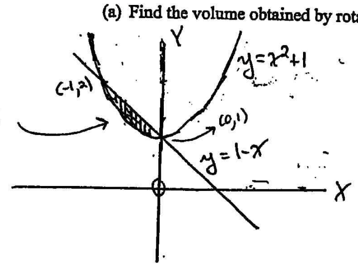
DEPARTMENT & COURSE NO: 136.170
EXAMINATION: Calculus II

TEST 2

TIME: 1 HOUR
EXAMINER: Various

[12] 3. Let S be the region in the plane enclosed by the curves $y=x^2+1$ and $y=1-x$.

(a) Find the volume obtained by rotating S around the x-axis.



The X-co-ordinates of the points of intersection satisfy

$$x^2+1 = 1-x$$
$$x^2+x = 0$$
$$x(x+1) = 0$$
$$x = 0 \text{ or } x = 1.$$

$\therefore$ the curves intersect at (-1,2) and (0,1). From the diagram, the line lies above that parabola if $-1 \le x \le 0$. The region S is shaded.

Using the "method of washers" we see that the volume V of the solid swept out by rotating S about the X-axis is given by:

$$V = \int_{-1}^{0} \pi (1-x)^2 \, dx - \int_{-1}^{0} \pi (x^2+1)^2 \, dx$$

$$= \pi \int_{-1}^{0} \left[(1-2x+x^2) - (x^4+2x^2+1)\right] dx$$

$$= \pi \int_{-1}^{0} (-2x - x^2 - x^4)\, dx = \pi \left(-x^2 - \frac{x^3}{3} - \frac{x^5}{5} \Big|_{-1}^{0} \right)$$

$$= \pi \left[0 - \left(-1 + \frac{1}{3} + \frac{1}{5}\right)\right] = \pi \left(\frac{15-5-3}{15} \right)$$

$$= \frac{7\pi}{15} \text{ cubic units}$$

continued...

(b) Find the volume obtained by rotating S around the y-axis.

Using the method of "cylindrical shells", we see that the volume V is given by:

$$V = \int_{-1}^{0} \underbrace{2\pi |x|}_{\text{radius of shell}} \underbrace{\left((1-x)-(x^2+1)\right)}_{\text{height of shell}} \underbrace{dx}_{\text{thickness of shell}}$$

As $x<0$ if $-1<x<0$, we see that $|x| = -x$ for the values of x we are using. Thus

$$V = 2\pi \int_{-1}^{0} (-x)\left[2-x-x^2\right]dx$$

$$= -2\pi \int_{-1}^{0} (2x - x^2 - x^3)\,dx = -2\pi \left(x^2 - \frac{x^3}{3} - \frac{x^4}{4} \Big|_{-1}^{0} \right)$$

$$= -2\pi \left(0 - \left(1 + \frac{1}{3} - \frac{1}{4}\right)\right) = 2\pi\left(1+\frac{1}{3}-\frac{1}{4}\right) = 2\pi\left(\frac{13}{12}\right)$$

$$= \frac{13\pi}{6} \text{ cubic units.}$$

[4] 4. Evaluate $\frac{d}{dx}\tan^{-1}(x^4)$ when $x=2$. Is your answer larger than $\frac{1}{8}$?

$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$ so $\frac{d}{dx}\left(\tan^{-1}(x^4)\right) = \frac{1}{1+(x^4)^2} \cdot \frac{d}{dx}(x^4)$

$= \frac{4x^3}{1+x^8}$ When $x=2$, this equals

$$\frac{4(2^3)}{1+2^8} = \frac{2^5}{1+2^8} = \frac{32}{257}.$$

But $\frac{2^5}{1+2^8} < \frac{2^5}{2^8} = \frac{1}{2^3} = \frac{1}{8}$, so the value is less than $\frac{1}{8}$.

continued...

[8] 5. Integrate, using any appropriate method.

(a) $\int_1^2 \frac{1}{x^2}\sin\left(\pi+\frac{\pi}{x}\right)dx$

Let $u = \pi + \frac{\pi}{x}$

$\therefore \frac{du}{dx} = \frac{-\pi}{x^2}$, so $\frac{dx}{x^2} = -\frac{1}{\pi}du$

When $x=1$, $u = \pi + \frac{\pi}{1} = 2\pi$. When $x=2$, $u = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$.

$\therefore \int_1^2 \frac{1}{x^2}\sin\left(\pi+\frac{\pi}{x}\right)dx = \int_{2\pi}^{\frac{3\pi}{2}} \sin(u)\left(-\frac{1}{\pi}du\right)$

$= \left(-\frac{1}{\pi}\right)\left(-\cos(u)\right)\Big|_{2\pi}^{\frac{3\pi}{2}} = \frac{1}{\pi}\left[\cos\left(\frac{3\pi}{2}\right) - \cos(2\pi)\right] = \frac{1}{\pi}\left[0-1\right] = -\frac{1}{\pi}$.

(b) $\int (2x-1)\,e^{-x}\,dx$

Let $u = 2x-1$ and $dv = e^{-x}dx$

$\therefore \frac{du}{dx} = 2$ so $du = 2dx$, and $v = \int e^{-x}dx = -e^{-x}$.

$\therefore \int (2x-1)e^{-x} = \int u\,dv = uv - \int v\,du$

$= (2x-1)(-e^{-x}) - \int(-e^{-x})(2\,dx)$

$= -e^{-x}(2x-1) + 2\int e^{-x}dx$

$= -e^{-x}(2x-1) - 2e^{-x} + C$.

THE END