Values

[10] 1. Find each limit, if it exists.

[4] (a) $\lim_{x\to 0} \frac{e^x - 1 + \sin x}{e^x + \ln(x+1) - 1}$

Solution.

$$\lim_{x\to 0} \frac{e^x - 1 + \sin x}{e^x + \ln(x+1) - 1} \overset{0/0-type}{=} \lim_{x\to 0} \frac{e^x + \cos x}{e^x + 1/(x+1)} = 1$$

[6] (b) $\lim_{x\to 0^+} (\sin x)^{\sin x}$

Solution.

Set $y = (\sin x)^{\sin x}$. Then $\ln y = \sin x \ln(\sin x)$. So

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sin x \ln(\sin x) = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{\sin x}} \overset{\frac{-\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cos x}{\frac{-1}{\sin^2 x}\cos x} = -\lim_{x\to 0^+} \sin x = 0.$$

It follows that $\ln\left(\lim_{x\to 0^+} y\right) = 0$, which gives $\lim_{x\to 0^+} y = e^0 = 1$

Values

[14] 2. Evaluate

[4] (a) $\int_{-1}^{1} x^3 \cos^8 x \, dx$

Denote $f(x) = x^3 \cos^8 x$. Then $f(-x) = ... = -f(x)$. So the function is odd. So the integral is 0.

[5] (b) $\int \frac{x^2 + \ln(2x)}{x} dx$

$\int \frac{x^2 + \ln(2x)}{x} dx = \int x\, dx + \int \frac{\ln(2x)}{x} dx = \frac{x^2}{2} + I + c_1$, where $I = \int \frac{\ln(2x)}{x} dx$ needs $u = \ln(2x)$; $du = \frac{1}{x} dx$; hence $I = \int \frac{\ln(2x)}{x} dx = \frac{u^2}{2} = \frac{(\ln(2x))^2}{2} + c_2$. Summarizing, $\int \frac{x^2 + \ln(2x)}{x} dx = \frac{x^2}{2} + \frac{(\ln(2x))^2}{2} + c$.

[5] (c) $\int_{0}^{\frac{\pi}{2}} \sin x \cos^8 x \, dx$

$\int_{0}^{\frac{\pi}{2}} \sin x \cos^8 x \, dx$ needs $u = \cos x$. The integral becomes

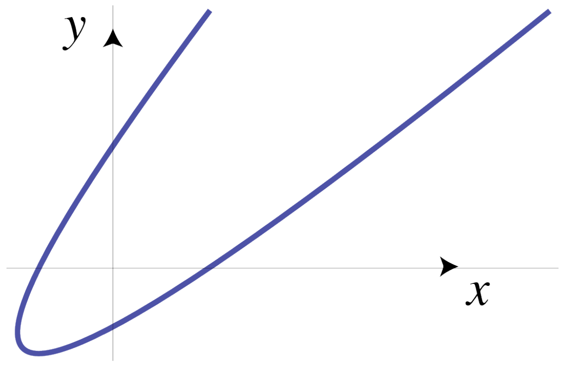
$$-\int_{x=0}^{x=\frac{\pi}{2}} u^8 \, du = -\frac{u^9}{9}\Bigg|_{x=0}^{x=\frac{\pi}{2}} = -\frac{\cos^9 x}{9}\Bigg|_{0}^{\frac{\pi}{2}} = \frac{1}{9}.$$

[6] 3. By using the definition of a definite integral write $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \cos\left(1 + i\frac{2}{n}\right) \sin^2\left(1 + i\frac{2}{n}\right)$ as a definite integral.

Solution. This is $\int_{1}^{3} \cos x \sin^2 x \, dx$.

Values

[10] 4. Consider the curve $x = t^2 + t - 1$, $y = t^2 - 1$. The graph of this curve is given below:



[5] (a) Find the point where the tangent line is horizontal, and find the points where the tangent line is vertical.

$\frac{dy}{dx} = \frac{2t}{2t+1}$. Horizontal tangent when $2t = 0$; the associated point is $(-1,-1)$. Vertical tangent when $2t+1=0$ or $t = -\frac{1}{2}$, giving the point $\left(-\frac{5}{4},-\frac{3}{4}\right)$.

[5] (b) Calculate the area of the region below the x-axis, bounded from below by the given curve from $t=0$ to $t=1$.

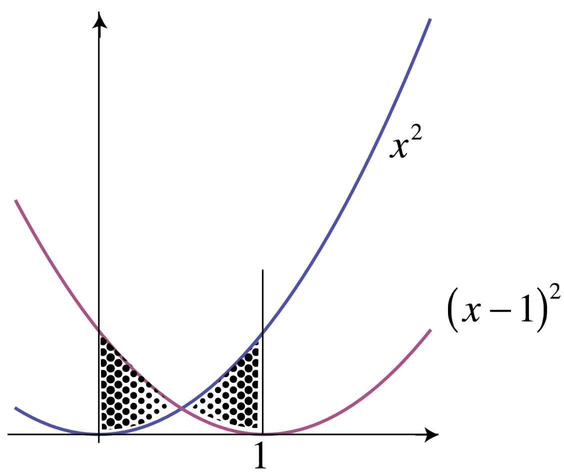
Solution

$$A = \int_{t=0}^{t=1} -y\,dx = -\int_0^1 (t^2-1)(2t+1)\,dt = -\int_0^1 (2t^3+t^2-2t-1)\,dt =$$

$$= -\left(\frac{t^4}{2} + \frac{t^3}{3} - t^2 - t\Big|_0^1\right) = -\left(\frac{1}{2}+\frac{1}{3}-2\right) = \frac{7}{6}.$$

Values

[10] 5. Calculate the area of the shaded region, bounded by the curves $y = x^2$, $y = (x-1)^2$, $x = 0$ and $x = 1$



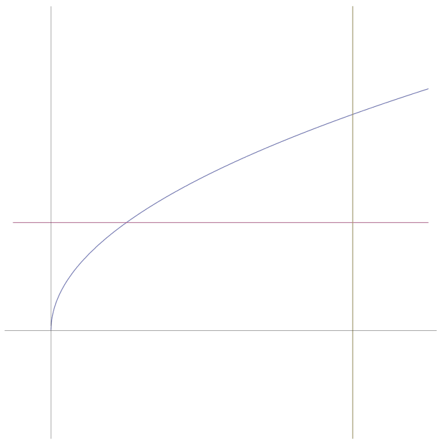
Solution. Solving $x^2 = (x-1)^2$; this is $2x - 1 = 0$ and so $x = \frac{1}{2}$.

So, the shaded region has area

$$\int_0^{\frac{1}{2}} \left((x-1)^2 - x^2\right)dx + \int_{\frac{1}{2}}^{1} \left(x^2 - (x-1)^2\right)dx = \int_0^{\frac{1}{2}} (1-2x)\,dx + \int_{\frac{1}{2}}^{1} (2x-1)\,dx =$$

$$= \left(x - x^2\right)\Big|_0^{\frac{1}{2}} + \left(x^2 - x\right)\Big|_{\frac{1}{2}}^{1} = \frac{1}{2} - \frac{1}{4} + (1-1) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{2}.$$

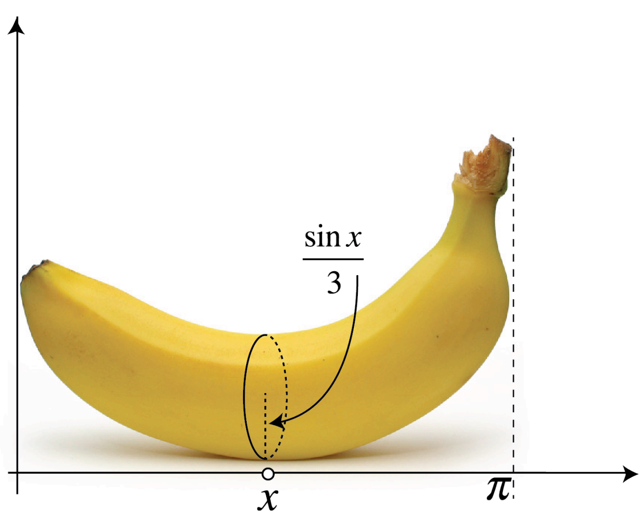
[10] [5] 6. (a) Find the volume of the solid obtained by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$. Sketch the region. Do NOT simplify your final answer.



Solving $\sqrt{x}=1$ gives $x=1$. So, the volume we want is

$$\int_1^4 (\sqrt{x}-1)^2 \pi\, dx = \pi\int_1^4 (x-2\sqrt{x}+1)\,dx = \pi\left(\frac{x^2}{2}-\frac{4}{3}x^{\frac{3}{2}}+x\right)\Big|_1^4 = \pi\left(\frac{16}{2}-\frac{4}{3}8+4\right)-\pi\left(\frac{1}{2}-\frac{4}{3}+1\right).$$

[5] (b) Consider the banana in the figure below; it is placed such that it extends from 0 to π along the x-axis. The cross section of the banana with a plane vertical to the x-axis and x units away from the origin is a disk of radius $\frac{\sin x}{3}$. Calculate the volume of the banana.



Solution.

$$V=\int_0^\pi \left(\frac{\sin x}{3}\right)^2 \pi\, dx = \frac{\pi}{9}\int_0^\pi \sin^2 x\, dx = \frac{\pi}{9}\int_0^\pi \frac{1}{2}(1-\cos 2x)\,dx = \frac{\pi}{18}\left(x-\frac{\sin 2x}{2}\right)\Big|_0^\pi = \frac{\pi^2}{18}.$$