Values

[16] 1. Find each limit, if it exists.

$(a) \quad \lim_{x\to 1} \frac{1-x+\ln x}{x^3-3x+2}$

Solution: $\lim_{x\to 1} \frac{1-x+\ln x}{x^3-3x+2} \overset{0/0}{\underset{H}{=}} \lim_{x\to 1} \frac{-1+\frac{1}{x}}{3x^2-3} \overset{0/0}{\underset{H}{=}} \lim_{x\to 1} \frac{-\frac{1}{x^2}}{6x} = -\frac{1}{6}$

$(b) \quad \lim_{x\to 0} \left(1+\sin(\pi x)\right)^{\frac{1}{x}}$

Solution. Set $y = \left(1+\sin(\pi x)\right)^{\frac{1}{x}}$. So $\ln y = \frac{\ln\left(1+\sin(\pi x)\right)}{x}$. Then

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln\left(1+\sin(\pi x)\right)}{x} \overset{0/0}{\underset{H}{=}} \lim_{x\to 0} \frac{\frac{1}{\left(1+\sin(\pi x)\right)}\left(\cos(\pi x)\right)\pi}{1} = -\pi$ and so

$\lim_{x\to 0} \left(1+\sin(\pi x)\right)^{\frac{1}{x}} = e^{-\pi}$.

[12] 2. Evaluate

$(a) \quad \int_0^1 x(2x-1)^{20}\, dx$

Solution. Use $u = 2x-1$ so that $du = 2dx$. Get:

$\int_0^1 x(2x-1)^{20}\, dx = \int_{-1}^1 \frac{u+1}{2}(u)^{20}\frac{du}{2} = \frac{1}{4}\left(\frac{u^{22}}{22}+\frac{u^{21}}{21}\right)\Bigg|_{-1}^{1} = \frac{1}{42}$.

$(b) \quad \int e^{\cos x} \sin x\, dx$

Solution. Use the substitution $u = \cos x$ and get $\int e^{\cos x}\sin x\, dx = -e^{\cos x} + c$.

[4] 3. (a) Use the definition of a definite integral to write $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[1+\left(\frac{i}{n}\right)^3\right]$ as a definite integral.

Solution. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[1+\left(\frac{i}{n}\right)^3\right] = \int_0^1 (1+x^3)dx$.

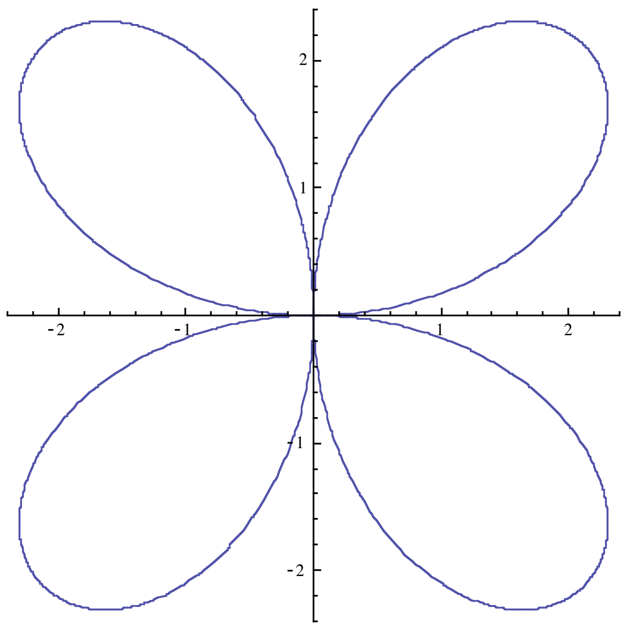
(b) Use your answer in part (a) to find the value of the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[1+\left(\frac{i}{n}\right)^3\right]$.

Solution. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[1+\left(\frac{i}{n}\right)^3\right] = \int_0^1 (1+x^3)dx = x + \frac{x^4}{4}\Bigg|_0^1 = \frac{5}{4}$.

Values

[9] 4. (a) Sketch the curve $r = 3\sin(2\theta)$. State the polar coordinates of at least three points on this graph.

Solution.

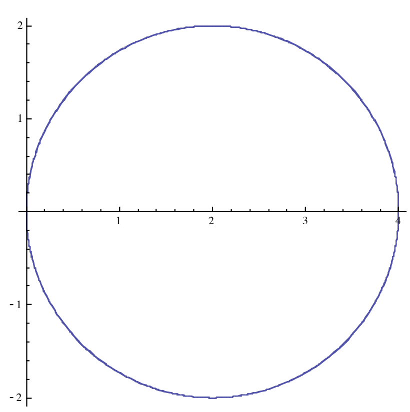


Three points: $(0,0)$, $(-3,\frac{3\pi}{4})$ and $(3,\frac{\pi}{4})$.

(b) Write the integral which represents the area of the region that lies inside the curve $r = 4\cos\theta$ and above the line $\theta = \frac{\pi}{4}$

Draw a rough sketch of the above two curves on the same coordinate system.]

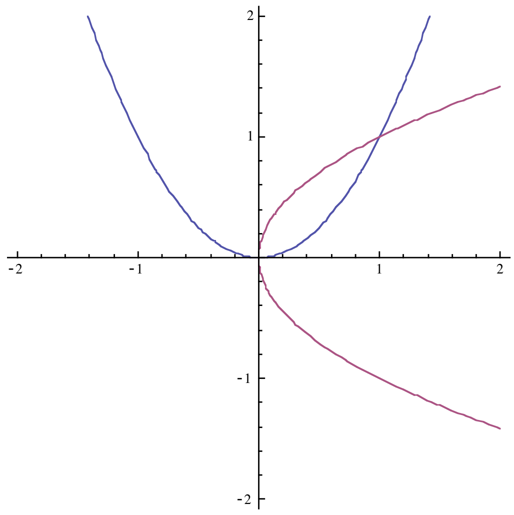
DO NOT EVALUATE THE INTEGRAL



Solution.

$$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(4\cos\theta)^2\, d\theta .$$

[8] 5. Find the area of the region bounded by the curves $y = x^2$ and $x = y^2$.
[Draw a rough sketch of the above two curves on the same coordinate system.]



The intersection points are $(0,0)$ and $(1,1)$.

The area of the region is $\int_0^2 (\sqrt{x} - x^2)dx = \frac{1}{3}$.

[11] 6. Write an integral which represents each volume.

(a) The volume of the solid obtained when the region bounded by the curve $y = \sin x$ with $0 \le x \le \pi$, and the x-axis is rotated about the y-axis.
DO NOT EVALUATE THIS INTEGRAL.

$$V = \int_0^{\pi} 2\pi x \sin x \; dx$$

(b) The volume of the solid obtained when the region enclosed by $y = x^2$ and $y = 2 - x^2$ is revolved about the x-axis.
DO NOT EVALUATE THIS INTEGRAL.

$$V = \int_{-1}^{1} \pi[(2 - x^2)^2 - (x^2)^2] \; dx .$$